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The Radiative Transport Equation with Waiting Time and its Diffusion Approximation with a Time-Fractional Derivative

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ABSTRACT

Albeit the past intensive research, the governing equation of anomalous diffusion which is observed for the transport of particles underground is still an open problem. In this paper, as a governing equation, the advection–diffusion equation with a time-fractional derivative term is derived from the radiative transport equation with an integral term for waiting time.

KEYWORDS

Diffusion approximation;
radiative transport equation;
time-fractional derivative

1. Introduction

Anomalous diffusion has been observed in different phenomena (Metzler and Klafter 2000). One example is mass transport in underground water (Adams and Gelhar 1992). As another example, an equation with an additional time-fractional term (Langlois, Farazmand, and Haller 2015) is derived from the fluid dynamics in the Basset problem (Basset 1888a,b,2015c, 1910; Boussinesq 1885). Aiming at reproducing experimental results for mass transport in porous media, various partial differential equations with fractional derivatives have been proposed. The fractional advection–dispersion equation with time- and spatial-fractional derivatives was proposed (Benson, Wheatcraft, and Meerschaert 2000). Fractional derivative models were compared (Sun et al. 2017; Wei, Chen, and Hon 2016) and parameters in fractional equations were estimated (Chakraborty, Meerschaert, and Lim 2009; Kelly et al. 2017). Orders of fractional derivatives were taken to be variables (Sun, Chen, and Chen 2009). Furthermore, the distributed order time fractional diffusion equation was tested (Liang et al. 2019). Trying to reproduce such anomalous diffusion, a multi-scaling

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tempered fractional-derivative model was proposed (Zhang et al. 2013). The tempered anomalous diffusion model, which has a time-fractional derivative, was used to analyze column flow experiment (Suzuki et al. 2016).

By the comparison of the numerical solution of the radiative transport equation and the observed concentration of tracer particles in a column experiment, it was found that the governing equation at the mesoscopic scale (i.e., the propagation distance, which is the column height, is comparable to the transport mean free path) is the radiative transport equation (Amagai, Hatano, and Machida 2021; Amagai et al. 2020).

In this paper, we show that an equation with a time-fractional derivative is asymptotically derived from the radiative transport equation with a waiting time. The derived equation governs transport in diffusive regime, in which the propagation distance is much larger than the transport mean free path.

The radiative transport equation, which is a linear Boltzmann equation, has been used in various fields. The diffusion approximation of the radiative transport equation has been intensively studied. In addition to the intuitive approach (Duderstadt and Martin 1979; Ishimaru 1978), The diffusion approximation was considered with the asymptotic expansion in the whole space (Ryzhik, Papanicolaou, and Keller 1996) and in the presence of the boundary (Larsen and Keller 1974). Bensoussan, Lions, and Papanicolaou explored the diffusion approximation in terms of homogenization (Bensoussan, Lions, and Papanicolaou 1979). The diffusion approximation in the presence of boundaries was treated for the propagation of elastic (seismic) waves (Bal 2001). Moreover, the Robin boundary condition, which is imposed on the diffusion equation, was considered with the diffusion approximation (Bardos, Santos, and Sentis 1984).

The remainder of the paper is organized as follows. In Section 2, the radiative transport equation with waiting time is introduced. In Section 3, the Laplace transform is taken to treat the integral term for traps. The diffusion approximation is considered in Section 4. Numerical solutions to the two equations which are connected by the diffusion approximation are compared in Section 5. Finally, concluding remarks are given in Section 6.

2. The radiative transport equation with waiting time

Let $\mathbf{v}_0 \in \mathbb{R}^3$ be the velocity of particles and $\mathbf{c} \in \mathbb{R}^3$ be advection. The direction of the propagation of a particle is given by $\hat{\boldsymbol{\theta}} = \mathbf{v}_0/|\mathbf{v}_0|$, which is a unit vector in $\mathbb{S}^2$. Let $\psi(\mathbf{r}, \hat{\boldsymbol{\theta}}, \tau)$ be the angular density of particles at position $\mathbf{r} \in \mathbb{R}^3$ in direction $\hat{\boldsymbol{\theta}} \in \mathbb{S}^2$ at time $\tau > 0$. The initial condition is imposed as

$$\psi(\mathbf{r}, \hat{\boldsymbol{\theta}}, 0) = a(\mathbf{r}, \hat{\boldsymbol{\theta}}),$$

where a is the initial distribution of particles. Let $\Phi(\tau)$ be the survival probability and $\sigma_a, \sigma_s, \sigma_{\text{trap}}$ be positive constants. We consider the following radiative transport equation in $\mathbb{R}^3$:

$$\begin{aligned} P_0 \psi(\mathbf{r}, \hat{\boldsymbol{\theta}}, \tau) + (\sigma_a + \sigma_s + \sigma_{\text{trap}}) \psi(\mathbf{r}, \hat{\boldsymbol{\theta}}, \tau) &= \sigma_s \int_{\mathbb{S}^2} p(\hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}}') \psi(\mathbf{r}, \hat{\boldsymbol{\theta}}', \tau) d\hat{\boldsymbol{\theta}}' \\ &+ \sigma_{\text{trap}} \int_0^\tau w(\tau - \tau') \psi(\mathbf{r}, \hat{\boldsymbol{\theta}}, \tau') d\tau' + \sigma_{\text{trap}} \Phi(\tau) a(\mathbf{r}, \hat{\boldsymbol{\theta}}), \end{aligned} \quad (1)$$

where

$$P_0 \psi(\mathbf{r}, \hat{\boldsymbol{\theta}}, \tau) = (\partial_\tau + (\mathbf{v}_0 + \mathbf{c}) \cdot \nabla) \psi(\mathbf{r}, \hat{\boldsymbol{\theta}}, \tau).$$

The scattering phase function satisfies $\int_{\mathbb{S}^2} p(\hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}}') d\hat{\boldsymbol{\theta}} = 1$. In this paper, σ_a, σ_s for absorption and scattering are assumed to be constant. The waiting-time distribution w is described below.

For column experiments, breakthrough curves were reproduced using (1) without σ_{trap} (Amagai, Hatano, and Machida 2021; Amagai et al. 2020). In the case of $\sigma_{\text{trap}} = 0$, we see that (1) governs mass transport if the particle flow in complicated paths is regarded as the transport of particles which undergo scattering (Amagai et al. 2020; Williams 1992a, 1992b, 1993a, 1993b). Although the column experiments were performed using uniformly random beads and sand, it can be expected that in field experiments tracer particles are trapped during the propagation or they propagate in an immobile zone (Hatano and Hatano 1998; Schumer et al. 2003). This motivates us to introduce terms with σ_{trap} in (1). The coefficient σ_{trap} on the left-hand side of (1) counts particles which are trapped and the integral term with σ_{trap} on the right-hand side means particles that were trapped at time τ' restart propagation at time τ .

A trapped particle reenters the flow after some waiting time or escape time. The function w in (1) is the waiting time distribution. If each particle immediately restarts the propagation even when it is trapped, the waiting-time function is $w(\tau) = \delta(\tau)$, where $\delta(\cdot)$ is Dirac's delta function. In this case, terms with σ_{trap} on both sides of (1) cancel. In general, $w(\tau)$ has a decaying behavior as τ grows. We have

$$\Phi(\tau) = 1 - \int_0^\tau w(\tau') d\tau'.$$

The term $\sigma_{\text{trap}} \Phi(\tau) a(\mathbf{r}, \hat{\boldsymbol{\theta}})$ on the right-hand side of (1) is the source term due to particles which are in dead-end pores at $\tau = 0$.

Recently, numerical simulation with dead-end pores was performed (Bordoloi et al. 2022). The probability density functions of escape time

obtained from trajectories of particles implies the power-law behavior of $w(\tau)$. Different power-law behaviors of the waiting time function were numerically observed for different shapes of dead-end pores (Hou, Jiang, and Wu 2018).

In light of these past researches, let us assume that w asymptotically decays as $\tau^{-(1+\alpha)}$, where $\alpha > 0$. Moreover we assume $0 < \alpha < 1$ for a long tail of w . Let $\gamma > 0$ be a constant. Examples of such function w include

$$w(\tau) = \frac{\alpha/\gamma}{(1 + \tau/\gamma)^{1+\alpha}}, \quad w(\tau) = \frac{\alpha(\gamma/\tau)^\alpha}{\tau(1 + (\gamma/\tau)^\alpha)^2}, \quad w(\tau) = \alpha\gamma^\alpha \tau^{-(1+\alpha)} e^{-(\gamma/\tau)^\alpha}.$$

In all cases, $\int_0^\infty w(\tau) d\tau = 1$. For large τ , they behave as

$$w(\tau) \sim \alpha\gamma^\alpha \tau^{-(1+\alpha)}.$$

Let us introduce a nondecreasing function $W(\tau)$ such that $dW/d\tau = w$ ($0 < \tau < \infty$), $W \in [0, 1]$, $W(0) = 0$. Corresponding to the above examples of w , we have

$$W(\tau) = 1 - \frac{1}{(1 + \tau/\gamma)^\alpha}, \quad W(\tau) = \frac{1}{1 + (\gamma/\tau)^\alpha}, \quad W(\tau) = e^{-(\gamma/\tau)^\alpha},$$

$$0 \leq \tau < \infty.$$

3. Laplace transform

Let us investigate the solution to (1) on a large time scale. To this end, we scale τ as

$$t = \epsilon\tau,$$

where $\epsilon > 0$ is a small number.

We write the solution of (1) as (Larsen and Keller 1974)

$$\psi(\mathbf{r}, \hat{\boldsymbol{\theta}}, \tau) = \psi^i(\mathbf{r}, \hat{\boldsymbol{\theta}}, \tau) + \psi^{\text{IL}}(\mathbf{r}, \hat{\boldsymbol{\theta}}, \tau),$$

where ψ^i is the interior part and ψ^{IL} is the initial layer part. In particular,

$$a(\mathbf{r}, \hat{\boldsymbol{\theta}}) = \psi^i(\mathbf{r}, \hat{\boldsymbol{\theta}}, 0) + \psi^{\text{IL}}(\mathbf{r}, \hat{\boldsymbol{\theta}}, 0).$$

The interior part $\psi^i(\mathbf{r}, \hat{\boldsymbol{\theta}}, \tau) = \psi_\epsilon^i(\mathbf{r}, \hat{\boldsymbol{\theta}}, t)$ satisfies

$$\begin{aligned} P_{0,\epsilon} \psi_\epsilon^i(\mathbf{r}, \hat{\boldsymbol{\theta}}, t) + (\sigma_a + \sigma_s + \sigma_{\text{trap}}) \psi_\epsilon^i(\mathbf{r}, \hat{\boldsymbol{\theta}}, t) \\ = \sigma_s \int_{\mathbb{S}^2} p(\hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}}') \psi_\epsilon^i(\mathbf{r}, \hat{\boldsymbol{\theta}}', t) d\hat{\boldsymbol{\theta}}' \\ + \frac{\sigma_{\text{trap}}}{\epsilon} \int_0^{t/\epsilon} w\left(\frac{t-t'}{\epsilon}\right) \psi_\epsilon^i(\mathbf{r}, \hat{\boldsymbol{\theta}}, t') dt' + \sigma_{\text{trap}} \Phi\left(\frac{t}{\epsilon}\right) a(\mathbf{r}, \hat{\boldsymbol{\theta}}), \end{aligned} \quad (2)$$

where

$$P_{0,\epsilon}\psi_\epsilon^i(\mathbf{r}, \hat{\boldsymbol{\theta}}, t) = (\epsilon\partial_t + (\mathbf{v}_0 + \mathbf{c}) \cdot \nabla)\psi_\epsilon^i(\mathbf{r}, \hat{\boldsymbol{\theta}}, t).$$

We write the Laplace transform of a function $f_\epsilon(t) = f(\tau)$ as

$$(\mathcal{L}f_\epsilon)(s) = \int_0^\infty e^{-st}f_\epsilon(t)dt, \quad s \in \mathbb{C}.$$

Let us take the Laplace transform for (2):

$$\begin{aligned} & (\mathcal{L}P_{0,\epsilon}\psi_\epsilon^i)(\mathbf{r}, \hat{\boldsymbol{\theta}}, s) + (\sigma_a + \sigma_s + \sigma_{\text{trap}})(\mathcal{L}\psi_\epsilon^i)(\mathbf{r}, \hat{\boldsymbol{\theta}}, s) \\ &= \sigma_s \int_{\mathbb{S}^2} p(\hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}}') (\mathcal{L}\psi_\epsilon^i)(\mathbf{r}, \hat{\boldsymbol{\theta}}', s) d\hat{\boldsymbol{\theta}}' \\ &+ \sigma_{\text{trap}}(\mathcal{L}w)(\epsilon s)(\mathcal{L}\psi_\epsilon^i)(\mathbf{r}, \hat{\boldsymbol{\theta}}, s) + \sigma_{\text{trap}} \frac{1 - (\mathcal{L}w)(\epsilon s)}{s} a(\mathbf{r}, \hat{\boldsymbol{\theta}}), \end{aligned}$$

where

$$(\mathcal{L}P_{0,\epsilon}\psi_\epsilon^i)(\mathbf{r}, \hat{\boldsymbol{\theta}}, s) = \epsilon s (\mathcal{L}\psi_\epsilon^i)(\mathbf{r}, \hat{\boldsymbol{\theta}}, s) - \epsilon\psi_\epsilon^i(\mathbf{r}, \hat{\boldsymbol{\theta}}, 0) + (\mathbf{v}_0 + \mathbf{c}) \cdot \nabla (\mathcal{L}\psi_\epsilon^i)(\mathbf{r}, \hat{\boldsymbol{\theta}}, s).$$

We set

$$d(\tau) = 1 - \left(\frac{\gamma}{\tau}\right)^\alpha.$$

We note that $W \sim d$ ($\tau \rightarrow \infty$) and for any $\lambda > 0$, $d(\lambda\tau)/d(\tau) \rightarrow 1$ as $\tau \rightarrow \infty$. We have for small s (Feller 1971; Karamata 1931),

$$(\mathcal{L}w)(s) \sim d\left(\frac{1}{s}\right) = 1 - (\gamma s)^\alpha, \quad s > 0. \quad (3)$$

Since $\epsilon > 0$ is small, we can replace $(\mathcal{L}w)(\epsilon s)$ with $1 - (\gamma\epsilon s)^\alpha$.

4. Diffusion approximation

Assuming slow advection, large scattering, small absorption, small trapping rate, and large life time, we introduce ϵ as

$$\mathbf{c} \rightarrow \epsilon\mathbf{c}, \quad \sigma_s \rightarrow \frac{\sigma_s}{\epsilon}, \quad \sigma_a \rightarrow \epsilon\sigma_a, \quad \sigma_{\text{trap}} \rightarrow \epsilon\sigma_{\text{trap}}, \quad \gamma \rightarrow \frac{\gamma}{\epsilon}.$$

Then we have

$$\begin{aligned} & \epsilon s (\mathcal{L}\psi_\epsilon^i)(\mathbf{r}, \hat{\boldsymbol{\theta}}, s) - \epsilon\psi_\epsilon^i(\mathbf{r}, \hat{\boldsymbol{\theta}}, 0) + (\mathbf{v}_0 + \epsilon\mathbf{c}) \cdot \nabla (\mathcal{L}\psi_\epsilon^i)(\mathbf{r}, \hat{\boldsymbol{\theta}}, s) \\ &+ \left(\epsilon\sigma_a + \frac{\sigma_s}{\epsilon} + \epsilon\sigma_{\text{trap}}\right) (\mathcal{L}\psi_\epsilon^i)(\mathbf{r}, \hat{\boldsymbol{\theta}}, s) = \frac{\sigma_s}{\epsilon} \int_{\mathbb{S}^2} p(\hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}}') (\mathcal{L}\psi_\epsilon^i)(\mathbf{r}, \hat{\boldsymbol{\theta}}', s) d\hat{\boldsymbol{\theta}}' \\ &+ \epsilon\sigma_{\text{trap}}(1 - (\gamma s)^\alpha) (\mathcal{L}\psi_\epsilon^i)(\mathbf{r}, \hat{\boldsymbol{\theta}}, s) + \epsilon\sigma_{\text{trap}}\gamma^\alpha s^{\alpha-1} a(\mathbf{r}, \hat{\boldsymbol{\theta}}), \end{aligned} \quad (4)$$

where we used (3). We will formally expand ψ^i as

$$\psi^i(\mathbf{r}, \hat{\boldsymbol{\theta}}, \tau) = \psi_\epsilon^i(\mathbf{r}, \hat{\boldsymbol{\theta}}, t) = \sum_{n=0}^{\infty} \epsilon^n \psi_n(\mathbf{r}, \hat{\boldsymbol{\theta}}, t).$$

Let us set $t_1 = \tau/\epsilon$. Since ψ^{IL} is responsible for the initial rapid change of ψ , we require ψ^{IL} to satisfy

$$(\partial_{t_1} + \epsilon(\mathbf{v}_0 + \epsilon \mathbf{c}) \cdot \nabla) \psi^{\text{IL}}(\mathbf{r}, \hat{\boldsymbol{\theta}}, \tau) = (\mathcal{K}_\epsilon - I)(\epsilon^2 \sigma_a + \sigma_s + \epsilon^2 \sigma_{\text{trap}}) \psi^{\text{IL}}(\mathbf{r}, \hat{\boldsymbol{\theta}}, \tau),$$

where I is the identity, and for a function $f(\hat{\boldsymbol{\theta}}, \tau)$, operator $\mathcal{K}_\epsilon$ is defined as

$$\begin{aligned} \mathcal{K}_\epsilon f(\hat{\boldsymbol{\theta}}, \tau) &= \frac{\sigma_s/\epsilon}{\epsilon \sigma_a + \sigma_s/\epsilon + \epsilon \sigma_{\text{trap}}} \int_{\mathbb{S}^2} p(\hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}}') f(\hat{\boldsymbol{\theta}}', \tau) d\hat{\boldsymbol{\theta}}' \\ &+ \frac{\epsilon \sigma_{\text{trap}}}{\epsilon \sigma_a + \sigma_s/\epsilon + \epsilon \sigma_{\text{trap}}} \int_0^\tau w(\tau - \tau') f(\hat{\boldsymbol{\theta}}, \tau') d\tau'. \end{aligned}$$

The initial condition is given by

$$\psi^{\text{IL}}(\mathbf{r}, \hat{\boldsymbol{\theta}}, 0) = a(\mathbf{r}, \hat{\boldsymbol{\theta}}) - \psi_0(\mathbf{r}, \hat{\boldsymbol{\theta}}, 0) - \sum_{n=1}^{\infty} \epsilon^n \psi_n(\mathbf{r}, \hat{\boldsymbol{\theta}}, 0).$$

In the limit of $\epsilon \rightarrow 0$, we have

$$\partial_{t_1} \psi^{\text{IL}}(\mathbf{r}, \hat{\boldsymbol{\theta}}, \tau) = (\mathcal{K}_0 - I) \sigma_s \psi^{\text{IL}}(\mathbf{r}, \hat{\boldsymbol{\theta}}, \tau),$$

where

$$\mathcal{K}_0 f(\hat{\boldsymbol{\theta}}, \tau) = \int_{\mathbb{S}^2} p(\hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}}') f(\hat{\boldsymbol{\theta}}', \tau) d\hat{\boldsymbol{\theta}}'.$$

The initial condition is written as

$$\psi^{\text{IL}}(\mathbf{r}, \hat{\boldsymbol{\theta}}, 0) = a(\mathbf{r}, \hat{\boldsymbol{\theta}}) - \psi_0(\mathbf{r}, \hat{\boldsymbol{\theta}}, 0).$$

The largest eigenvalue of $\mathcal{K}_0$ is 1 (Dautray and Lions 1993; Larsen and Keller 1974). Hence, ψ^{IL} does not vanish if the initial value is nonzero and has a component proportional to the corresponding eigenfunction, which is independent of $\hat{\boldsymbol{\theta}}$. This implies

$$\psi_0(\mathbf{r}, \hat{\boldsymbol{\theta}}, 0) = \frac{1}{4\pi} \int_{\mathbb{S}^2} a(\mathbf{r}, \hat{\boldsymbol{\theta}}) d\hat{\boldsymbol{\theta}}.$$

Thus, ψ^{IL} decays exponentially in time. Since $t_1 = \epsilon^{-2}t$, the decay rate in t is very large, being proportional to ϵ^{-2} . The initial layer part ψ^{IL} is vanishingly small outside the initial layer of duration $O(\epsilon^2)$.

By considering terms of order $O(\epsilon^{-1})$ in (4), we have

$$(\mathcal{L}\psi_0)(\mathbf{r}, \hat{\boldsymbol{\theta}}, s) = \int_{\mathbb{S}^2} p(\hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}}') (\mathcal{L}\psi_0)(\mathbf{r}, \hat{\boldsymbol{\theta}}', s) d\hat{\boldsymbol{\theta}}'.$$

The above relation implies that ψ_0 is independent of $\hat{\boldsymbol{\theta}}$ and we can write

$$\psi_0(\mathbf{r}, \hat{\boldsymbol{\theta}}, t) = \frac{1}{4\pi} u(\mathbf{r}, t).$$

We have

$$u(\mathbf{r}, 0) = a_0(\mathbf{r}),$$

where

$$a_0(\mathbf{r}) = \int_{\mathbb{S}^2} a(\mathbf{r}, \hat{\boldsymbol{\theta}}) d\hat{\boldsymbol{\theta}}.$$

Next we collect terms of order $O(1)$:

$$\mathbf{v}_0 \cdot \nabla (\mathcal{L}\psi_0)(\mathbf{r}, \hat{\boldsymbol{\theta}}, s) + \sigma_s (\mathcal{L}\psi_1)(\mathbf{r}, \hat{\boldsymbol{\theta}}, s) = \sigma_s \int_{\mathbb{S}^2} p(\hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}}') (\mathcal{L}\psi_1)(\mathbf{r}, \hat{\boldsymbol{\theta}}', s) d\hat{\boldsymbol{\theta}}'.$$

Thus,

$$\psi_1(\mathbf{r}, \hat{\boldsymbol{\theta}}, t) = -\frac{|\mathbf{v}_0|}{4\pi(1-g)\sigma_s} \hat{\boldsymbol{\theta}} \cdot \nabla u(\mathbf{r}, t),$$

where

$$g = \int_{\mathbb{S}^2} \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}}' p(\hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}}') d\hat{\boldsymbol{\theta}}'.$$

Terms of order $O(\epsilon)$ satisfies the relation below:

$$\begin{aligned} & s(\mathcal{L}\psi_0)(\mathbf{r}, \hat{\boldsymbol{\theta}}, s) - \psi_0(\mathbf{r}, \hat{\boldsymbol{\theta}}, 0) + \mathbf{v}_0 \cdot \nabla (\mathcal{L}\psi_1)(\mathbf{r}, \hat{\boldsymbol{\theta}}, s) + \mathbf{c} \cdot \nabla (\mathcal{L}\psi_0)(\mathbf{r}, \hat{\boldsymbol{\theta}}, s) \\ & + \sigma_a (\mathcal{L}\psi_0)(\mathbf{r}, \hat{\boldsymbol{\theta}}, s) + \sigma_s (\mathcal{L}\psi_2)(\mathbf{r}, \hat{\boldsymbol{\theta}}, s) = \sigma_s \int_{\mathbb{S}^2} p(\hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}}') (\mathcal{L}\psi_2)(\mathbf{r}, \hat{\boldsymbol{\theta}}', s) d\hat{\boldsymbol{\theta}}' \\ & - \sigma_{\text{trap}} (\gamma s)^\alpha (\mathcal{L}\psi_0)(\mathbf{r}, \hat{\boldsymbol{\theta}}, p) + \sigma_{\text{trap}} \gamma^\alpha s^{\alpha-1} a(\mathbf{r}, \hat{\boldsymbol{\theta}}). \end{aligned}$$

By integrating the above equation over $\hat{\boldsymbol{\theta}}$, we obtain

$$\begin{aligned} & s(\mathcal{L}u)(\mathbf{r}, s) - u(\mathbf{r}, 0) - \frac{|\mathbf{v}_0|^2}{3(1-g)\sigma_s} \Delta(\mathcal{L}u)(\mathbf{r}, s) + \mathbf{c} \cdot \nabla (\mathcal{L}u)(\mathbf{r}, s) + \sigma_a (\mathcal{L}u)(\mathbf{r}, s) \\ & = -\sigma_{\text{trap}} (\gamma s)^\alpha (\mathcal{L}u)(\mathbf{r}, p) + \sigma_{\text{trap}} \gamma^\alpha s^{\alpha-1} a_0(\mathbf{r}). \end{aligned} \tag{5}$$

Let us set

$$\eta = \gamma^\alpha \sigma_{\text{trap}}, \quad D_0 = \frac{|\mathbf{v}_0|^2}{3(1-g)\sigma_s}.$$

We can rewrite (5) as

$$s(\mathcal{L}u)(\mathbf{r}, s) - u(\mathbf{r}, 0) - D_0 \Delta(\mathcal{L}u)(\mathbf{r}, s) + \mathbf{c} \cdot \nabla(\mathcal{L}u)(\mathbf{r}, s) + \sigma_a(\mathcal{L}u)(\mathbf{r}, s) + \eta(s^\alpha(\mathcal{L}u)(\mathbf{r}, s) - s^{\alpha-1}a_0(\mathbf{r})) = 0. \quad (6)$$

For a function $f(t)$, the Caputo derivative ∂_t^α ($0 < \alpha < 1$) is given by Caputo (1967), Podlubny (1999)

$$\partial_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-t')^{-\alpha} \partial_{t'} f(t') dt', \quad 0 < \alpha < 1. \quad (7)$$

Here, $\Gamma(\cdot)$ is the gamma function. Then we have

$$\mathcal{L}(\partial_t^\alpha f)(s) = s^\alpha(\mathcal{L}f)(s) - s^{\alpha-1}f(0). \quad (8)$$

When defined in (7), the function f needs to satisfy certain conditions (e.g., f has to be differentiable); see Kubica, Ryszewska, and Yamamoto (2020) for the mathematical treatment of the Caputo derivative. We assume that $\partial_t^\alpha f$ exists.

Finally, we obtain

$$\begin{cases} \partial_t u(\mathbf{r}, t) + \eta \partial_t^\alpha u(\mathbf{r}, t) - D_0 \Delta u(\mathbf{r}, t) + \mathbf{c} \cdot \nabla u(\mathbf{r}, t) + \sigma_a u(\mathbf{r}, t) = 0, & \mathbf{r} \in \mathbb{R}^3, \quad t > 0, \\ u(\mathbf{r}, 0) = a_0(\mathbf{r}), & \mathbf{r} \in \mathbb{R}^3. \end{cases} \quad (9)$$

5. Numerical test

Let us suppose that the advection $\mathbf{c}$ is negligibly small and the speed $|\mathbf{v}_0|$ is 1. Moreover, let us consider the isotropic scattering of $p = 1/(4\pi)$. To see the mathematical structure, we consider the following simple one-dimensional case:

$$\begin{cases} (\partial_t + \mu \partial_x + \sigma_a + \sigma_s + \sigma_{\text{trap}}) \psi(x, \mu, t) = \frac{\sigma_s}{2} \int_{-1}^1 \psi(x, \mu, t) d\mu \\ + \sigma_{\text{trap}} \int_0^t w(t-t') \psi(x, \mu, t') dt' + \sigma_{\text{trap}} \Phi(t) a(x, \mu), \\ -\infty < x < \infty, \quad -1 \leq \mu \leq 1, \quad t > 0, \\ \psi(x, \mu, 0) = a(x, \mu), \quad -\infty < x < \infty, \quad -1 \leq \mu \leq 1, \end{cases} \quad (10)$$

where μ is the cosine of the polar angle. To be specific, we set

$$a(x, \mu) = \delta(x), \quad w(\tau) = \frac{\alpha/\gamma}{(1 + \tau/\gamma)^{1+\alpha}}, \quad 0 < \alpha < 1, \quad \gamma > 0.$$

We note that in this case

$$(\mathcal{L}w)(s) = \alpha e^{\gamma s} E_{1+\alpha}(\gamma s),$$

where $E_{1+\alpha}$ is the generalized exponential integral: $E_{1+\alpha}(z) = z^\alpha \int_z^\infty e^{-t}/t^{1+\alpha} dt$, $z \in \mathbb{C} \setminus (-\infty, 0]$.

Let $(\mathcal{L}\psi)(x, \mu, s)$ be the Laplace transform of $\psi(x, \mu, t)$. Here, $\mathcal{L}\psi$ satisfies

$$(\mu\partial_x + \sigma_t)(\mathcal{L}\psi)(x, \mu, s) = \frac{\sigma_s}{2} \int_{-1}^1 (\mathcal{L}\psi)(x, \mu, s) d\mu + q(x, s)$$

for $-\infty < x < \infty$, $-1 \leq \mu \leq 1$, where

$$\begin{aligned} q(x, s) &= (\sigma_{\text{trap}}(\mathcal{L}\Phi)(s) + 1)\delta(x), \\ \sigma_t(s) &= \sigma_a + \sigma_s + (\sigma_{\text{trap}}(\mathcal{L}\Phi)(s) + 1)s. \end{aligned}$$

We note that the relation $(\mathcal{L}\Phi)(s) = [1 - (\mathcal{L}w)(s)]/s$ was used for the last line of the above equation.

By employing the analytical discrete ordinates method (ADO) (Barichello 2011; Barichello, Garcia, and Siewert 2000; Barichello et al. 2002; Barichello and Siewert 1999a,b; Siewert and Wright 1999), we approximate $\mathcal{L}\psi$ with v which satisfies

$$(\mu_i\partial_x + \sigma_t(s))v(x, \mu_i, s) = \frac{\sigma_s}{2} \sum_{j=1}^N w_j [v(x, \mu_j, s) + v(x, -\mu_j, s)] + q(x, s)$$

for $-\infty < x < \infty$. Here, μ_i, w_i ($i = 1, 2, \dots, N$) are abscissas and weights for the Gauss–Legendre quadrature, respectively. We label μ_i as $0 < \mu_1 < \dots < \mu_N < 1$. We can write

$$v(x, \mu_i, s) = (\sigma_{\text{trap}}(\mathcal{L}\Phi)(s) + 1) \sum_{j=1}^N [G(x, \mu_i; 0, \mu_j; s) + G(x, \mu_i; 0, -\mu_j; s)].$$

Here, the fundamental solution $G(x, \mu_i; y, \mu_j; s)$ satisfies

$$\begin{aligned} (\mu_i\partial_x + \sigma_t(s))G(x, \mu_i; y, \mu_j; s) &= \frac{\sigma_s}{2} \sum_{j=1}^N w_j [G(x, \mu_i; y, \mu_j; s) + G(x, \mu_i; y, -\mu_j; s)] \\ &\quad + \delta(x - y)\delta_{ij}, \end{aligned}$$

where δ_{ij} is the Kronecker delta.

To obtain $G(x, \mu_i; y, \mu_j; s)$, we consider the homogeneous equation:

$$(\mu_i\partial_x + \sigma_t(s))v_\nu(x, \mu_i, s) = \frac{\sigma_s}{2} \sum_{j=1}^N w_j [v_\nu(x, \mu_j, s) + v_\nu(x, -\mu_j, s)]$$

with the separation constant ν , which implicitly depends on s . We write v_ν as

$$v_\nu(x, \mu_i, s) = \phi(\nu, \mu_i, s)e^{-x/\nu}.$$

We impose the normalization condition:

$$\sum_{i=1}^N w_i (\phi(\nu, \mu_i, s) + \phi(\nu, -\mu_i, s)) = 1.$$

We note the relation $\phi(-\nu, \mu_i, s) = \phi(\nu, -\mu_i, s)$. Assuming that $\nu \neq \mu_i/\sigma_t(s)$, we obtain

$$\phi(\nu, \mu_i, s) = \frac{\sigma_s \nu}{2} \frac{1}{\sigma_t(s) \nu - \mu_i}. \quad (11)$$

The following orthogonality relation holds:

$$\sum_{i=1}^N w_i \mu_i [\phi(\nu, \mu_i, s) \phi(\nu', \mu_i, s) - \phi(\nu, -\mu_i, s) \phi(\nu', -\mu_i, s)] = \mathcal{N}(\nu, s) \delta_{\nu, \nu'},$$

where

$$\mathcal{N}(\nu, s) = \sum_{i=1}^N w_i \mu_i [\phi(\nu, \mu_i, s)^2 - \phi(\nu, -\mu_i, s)^2].$$

To compute ν , we introduce

$$\Xi_{\pm} = \begin{pmatrix} \pm \mu_1 & & & \\ & \pm \mu_2 & & \\ & & \ddots & \\ & & & \pm \mu_N \end{pmatrix}, \quad \Phi_{\pm}(\nu, s) = \begin{pmatrix} \phi(\nu, \pm \mu_1, s) \\ \phi(\nu, \pm \mu_2, s) \\ \vdots \\ \phi(\nu, \pm \mu_N, s) \end{pmatrix}, \quad \{W\}_{ij} = w_j.$$

Then the homogeneous equation can be written as

$$\begin{aligned} \left(\sigma_t(s) I_N - \frac{1}{\nu} \Xi_{+} \right) \Phi_{+}(\nu, s) &= \frac{\sigma_s}{2} W(\Phi_{+}(\nu, s) + \Phi_{-}(\nu, s)), \\ \left(\sigma_t(s) I_N - \frac{1}{\nu} \Xi_{-} \right) \Phi_{-}(\nu, s) &= \frac{\sigma_s}{2} W(\Phi_{+}(\nu, s) + \Phi_{-}(\nu, s)), \end{aligned}$$

where I_N is the identity matrix of dimension N . The above equation can be rewritten as

$$\begin{pmatrix} \Xi_{+}^{-1} & \\ & \Xi_{-}^{-1} \end{pmatrix} \left[\sigma_t(s) \begin{pmatrix} I_N & \\ & I_N \end{pmatrix} - \frac{\sigma_s}{2} \begin{pmatrix} W & W \\ W & W \end{pmatrix} \right] \begin{pmatrix} \Phi_{+}(\nu, s) \\ \Phi_{-}(\nu, s) \end{pmatrix} = \frac{1}{\nu} \begin{pmatrix} \Phi_{+}(\nu, s) \\ \Phi_{-}(\nu, s) \end{pmatrix}.$$

The fact $\Phi_{\pm}(-\nu, s) = \Phi_{\mp}(\nu, s)$ implies that $-\nu$ is an eigenvalue if ν is an eigenvalue. Thus, $\nu = \pm \nu_n(s)$ ($n = 1, 2, \dots, N$) are computed as eigenvalues of the above eigenproblem ($\nu_n(s) > 0$ assuming $\nu_n(s)$ are nonzero). The eigenvectors of the eigenproblem are not necessary since $\phi(\nu, \mu_i, s)$ are already explicitly given in (11). We note that, unlike the standard ADO for the equation with real coefficients, in the present case $\nu_n(s)$ are complex due to $\sigma_t(s) \in \mathbb{C}$. The fundamental solution is given by

$$G(x, \mu_i; y, \mu_j, s) = \pm \sum_{\pm \Re \nu_n > 0} \frac{w_j}{\mathcal{N}(\nu_n, s)} \phi(\nu_n, \mu_i, s) \phi(\nu_n, \mu_j, s) e^{-(x-y)/\nu_n(s)},$$

where upper signs are chosen for $x > y$ and lower signs are used for $x < y$.

Let us consider

$$U_{\text{RTE}}(x, t; \alpha) = \mathcal{L}^{-1} \sum_{i=1}^N w_i [v(x, \mu_i, s) + v(x, -\mu_i, s)], \quad x > 0, \quad t > 0,$$

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform. We have

$$(\mathcal{L}U_{\text{RTE}})(x, s; \alpha) = (\sigma_{\text{trap}}(\mathcal{L}\Phi)(s) + 1) \sum_{\Re \nu_n > 0} \frac{1}{\mathcal{N}(\nu_n, s)} e^{-x/\nu_n(s)}.$$

The particle density $U_{\text{RTE}}(x, t; \alpha)$ is given by

$$\begin{aligned} U_{\text{RTE}}(x, t; \alpha) &= \mathcal{L}^{-1}(\mathcal{L}U_{\text{RTE}})(x, t; \alpha) \\ &= \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{st} (\mathcal{L}U_{\text{RTE}})(x, s; \alpha) ds, \end{aligned}$$

where $\sigma \in \mathbb{R}$ is taken to be greater than the largest real part of any singularity. Assuming that $U_{\text{RTE}}(x, t)$ is real, we have

$$\Re(\mathcal{L}U_{\text{RTE}})(x, \sigma + i\omega; \alpha) = \int_0^\infty e^{-\sigma t} U_{\text{RTE}}(x, t; \alpha) \cos(\omega t) dt, \quad \omega \in \mathbb{R}.$$

By integrating both sides of the above equation over ω after multiplying $(2e^{\sigma t}/\pi) \cos \omega t$ ($t > 0$), we obtain (Davies and Martin 1979; Ganapol 2008)

$$U_{\text{RTE}}(x, t; \alpha) = \frac{2e^{\sigma t}}{\pi t} \int_0^\infty \Re(\mathcal{L}U_{\text{RTE}}) \left(x, \sigma + i\frac{\omega}{t}; \alpha \right) \cos \omega \, d\omega. \quad (12)$$

The inverse Laplace transform with (12) can be performed numerically. We will use the double-exponential formula (Ooura and Mori 1991, 1999). Let us introduce

$$\varphi(y) = \frac{y}{1 - e^{-K \sinh y}}, \quad K = 6.$$

Here, $K = 6$ was found by numerical experiments. Then the above integral can be expressed as

$$U_{\text{RTE}}(x, t; \alpha) = \frac{2e^{\sigma t}}{\pi t} \int_{-\infty}^\infty \cos(M\varphi(y)) \Re(\mathcal{L}U_{\text{RTE}}) \left(x, \sigma + \frac{iM}{t} \varphi(y); \alpha \right) M \frac{d\varphi(y)}{dy} dy,$$

where $\omega = M\varphi(y)$, $M > 0$. We introduce $h = \pi/M$. The above integral can be numerically computed with the trapezoidal rule (Sugihara 1997; Trefethen and Weideman 2014). We have

$$U_{\text{RTE}}(x, t; \alpha) \approx \frac{2e^{\sigma t}}{t} \sum_{j=-J}^J \cos(M\varphi(y_j)) \Re(\mathcal{L}U_{\text{RTE}})\left(x, \sigma + \frac{iM}{t}\varphi(y_j); \alpha\right) \frac{d\varphi(y)}{dy} \Big|_{y_j},$$

where J is an integer and $y_j = jh + \pi/(2M)$. We note that $d\varphi/dy \rightarrow 0$ double-exponentially as $y \rightarrow -\infty$. We see that $\varphi(y) \rightarrow y$ double exponentially as $y \rightarrow \infty$ and $\cos(M\varphi(jh + \pi/(2M))) \sim \cos(Mjh + \pi/2) = 0$ for large j .

For the numerical calculation, we set $\alpha = 1/2$, $\sigma = 0.04$, $\sigma_a = 10^{-9} \text{ min}^{-1}$, $\sigma_s = 1 \text{ min}^{-1}$, $N = 30$, $k_{\max} = 40$, $M = 40$.

Next we consider the one-dimensional version of (9):

$$\begin{cases} (\partial_t + \eta \partial_t^\alpha) U_{\text{DE}}(x, t; \alpha) - D_0 \partial_x^2 U_{\text{DE}}(x, t; \alpha) + \sigma_a U_{\text{DE}}(x, t; \alpha) = 0, & x \in \mathbb{R}, \quad t > 0, \\ U_{\text{DE}}(x, 0; \alpha) = a_0(x), & x \in \mathbb{R}, \end{cases} \quad (13)$$

where the advection term was ignored ($\mathbf{c} = \mathbf{0}$) and

$$a_0(x) = 2\delta(x).$$

We note that $D_0 = 1/(3\sigma_s)$. The function

$$V(k, s) = \int_{-\infty}^{\infty} e^{-ikx} (\mathcal{L}U_{\text{DE}})(x, s; \alpha) dx, \quad -\infty < k < \infty$$

satisfies

$$V(k, s) = \frac{1 + \eta s^{\alpha-1}}{s + \eta s^\alpha + D_0 k^2 + \sigma_a} \int_{-\infty}^{\infty} e^{-ikx} a_0(x) dx$$

Since $\int_0^\infty e^{-by} dy = 1/b$, we have

$$V(k, s) = 2(1 + \eta s^{\alpha-1}) \int_0^\infty e^{-(s + \eta s^\alpha + D_0 k^2 + \sigma_a)y} dy.$$

By a straightforward calculation we can show

$$\mathcal{L}\Theta(t-y)g_\alpha\left(\frac{t-y}{(\eta y)^{1/\alpha}}\right)(\eta y)^{-1/\alpha} = e^{-sy}(\mathcal{L}g_\alpha)\left((\eta y)^{1/\alpha}s\right) = e^{-(s+\eta s^\alpha)y},$$

where $\Theta(\cdot)$ is the Heaviside step function and $g_\alpha(\cdot)$ is such that $\mathcal{L}g_\alpha = e^{-s^\alpha}$. We have

$$\mathcal{L}^{-1}e^{-s^\alpha} = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \mathcal{L}^{-1}s^{\alpha j} = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{t^{-\alpha j-1}}{\Gamma(-\alpha j)} = \frac{1}{t} F_\alpha\left(\frac{1}{t^\alpha}\right) = \frac{\alpha}{t^{1+\alpha}} M_\alpha\left(\frac{1}{t^\alpha}\right),$$

where using the Mainardi–Wright function $W_{\lambda, \mu}(z)$,

$$F_\alpha(z) = W_{-\alpha, 0}(-z), \quad M_\alpha(z) = W_{-\alpha, 1-\alpha}(-z), \quad 0 < \alpha < 1.$$

Hence (Mainardi, Mura, and Pagnini 2010),

$$g_\alpha(t) = \frac{1}{t} F_\alpha\left(\frac{1}{t^\alpha}\right) = \frac{\alpha}{t^{1+\alpha}} M_\alpha\left(\frac{1}{t^\alpha}\right).$$

In particular, we have

$$M_{1/2}(z) = \frac{1}{\sqrt{\pi}} e^{-z^2/4}.$$

We note that (Schumer et al. 2003)

$$\lim_{\eta \rightarrow 0} g_\alpha\left(\frac{t-y}{(\eta y)^{1/\alpha}}\right) (\eta y)^{-1/\alpha} = \lim_{\eta \rightarrow 0} \frac{\alpha \eta y}{(t-y)^{1+\alpha}} M_\alpha\left(\frac{\eta y}{(t-y)^\alpha}\right) = \delta(t-y). \quad (14)$$

Thus,

$$\begin{aligned} & (\mathcal{L}U_{\text{DE}})(x, s; \alpha) \\ &= \frac{1 + \eta s^{\alpha-1}}{\pi} \int_0^\infty \left(\mathcal{L}\Theta(t-y) g_\alpha\left(\frac{t-y}{(\eta y)^{1/\alpha}}\right) (\eta y)^{-1/\alpha} \right) \int_{-\infty}^\infty e^{ikx} e^{-(D_0 k^2 + \sigma_a)y} dk dy \\ &= \frac{1}{\pi} \mathcal{L}\left(\delta(t) + \frac{\eta t^{-\alpha}}{\Gamma(1-\alpha)}\right)(s) \\ &\quad \times \int_0^\infty \left(\mathcal{L}\Theta(t-y) g_\alpha\left(\frac{t-y}{(\eta y)^{1/\alpha}}\right) (\eta y)^{-1/\alpha} \right) \sqrt{\frac{\pi}{y D_0}} e^{-\frac{x^2}{4y D_0}} e^{-\sigma_a y} dy, \end{aligned}$$

where we used under the condition $\Re s > 0$,

$$1 + \eta s^{\alpha-1} = (\mathcal{L}\delta(t))(s) + \frac{\eta}{\Gamma(1-\alpha)} (\mathcal{L}t^{-\alpha})(s).$$

By the inverse Laplace transform we obtain

$$\begin{aligned} U_{\text{DE}}(x, t; \alpha) &= \frac{1}{\eta^{1/\alpha} \sqrt{\pi D_0}} \int_0^t \left(\delta(t-t') + \frac{\eta(t-t')^{-\alpha}}{\Gamma(1-\alpha)} \right) \\ &\quad \times \left(\int_0^{t'} g_\alpha\left(\frac{t'-y}{(\eta y)^{1/\alpha}}\right) y^{-\frac{1}{\alpha}-\frac{1}{2}} e^{-\frac{x^2}{4y D_0}} e^{-\sigma_a y} dy \right) dt' \\ &= \frac{\alpha \eta}{\sqrt{\pi D_0}} \int_0^t \left(\delta(t-t') + \frac{\eta(t-t')^{-\alpha}}{\Gamma(1-\alpha)} \right) \\ &\quad \times \left(\int_0^{t'} \frac{\sqrt{y}}{(t'-y)^{1+\alpha}} M_\alpha\left(\frac{\eta y}{(t'-y)^\alpha}\right) e^{-\frac{x^2}{4y D_0}} e^{-\sigma_a y} dy \right) dt'. \end{aligned}$$

In the limit of $\sigma_{\text{trap}} \rightarrow 0$, we recover the normal diffusion:

$$\lim_{\eta \rightarrow 0} U_{\text{DE}}(x, t; \alpha) = \frac{1}{\sqrt{\pi D_0}} t^{-\frac{1}{2}} e^{-\frac{x^2}{4t D_0}} e^{-\sigma_a t}, \quad (15)$$

where the relation (14) was used.

In the case of $\alpha = 1/2$, we obtain

$$U_{\text{DE}}\left(x, t; \frac{1}{2}\right) = \frac{\eta}{2\pi\sqrt{D_0}} \int_0^t \left(\delta(t-t') + \frac{\eta}{\sqrt{\pi}(t-t')^{1/2}} \right) \\ \times \left(\int_0^{t'} \frac{\sqrt{y}}{(t'-y)^{3/2}} e^{-\frac{\eta^2 y^2}{4(t'-y)}} e^{-\frac{x^2}{4D_0 y}} e^{-\sigma_a y} dy \right) dt',$$

where we used $\Gamma(1/2) = \sqrt{\pi}$. For a numerical purpose, we change variables as $y = tt_1$ and then change $t_1 = 1 - \tau_1^2$ in the first term. Similarly we change variables as $t' = tt_1$, $y = tt_1 t_2$ and then change t_2 as $t_2 = 1 - \tau_2^2$ in the second term. We obtain

$$U_{\text{DE}}\left(x, t; \frac{1}{2}\right) = \frac{\eta}{\pi\sqrt{D_0}} \int_0^1 \frac{\sqrt{1-\tau_1^2}}{\tau_1^2} e^{-\frac{\eta^2 t(1-\tau_1^2)^2}{4\tau_1^2}} e^{-\frac{x^2}{4D_0 t(1-\tau_1^2)}} e^{-\sigma_a t(1-\tau_1^2)} d\tau_1 \\ + \frac{\eta^2 \sqrt{t}}{\pi\sqrt{\pi D_0}} \int_0^1 \frac{1}{\sqrt{1-\tau_1}} \\ \times \left(\int_0^1 \frac{\sqrt{1-\tau_2^2}}{\tau_2^2} e^{-\frac{\eta^2 t t_1 (1-\tau_2^2)^2}{4\tau_2^2}} e^{-\frac{x^2}{4D_0 t t_1 (1-\tau_2^2)}} e^{-\sigma_a t t_1 (1-\tau_2^2)} d\tau_2 \right) dt_1.$$

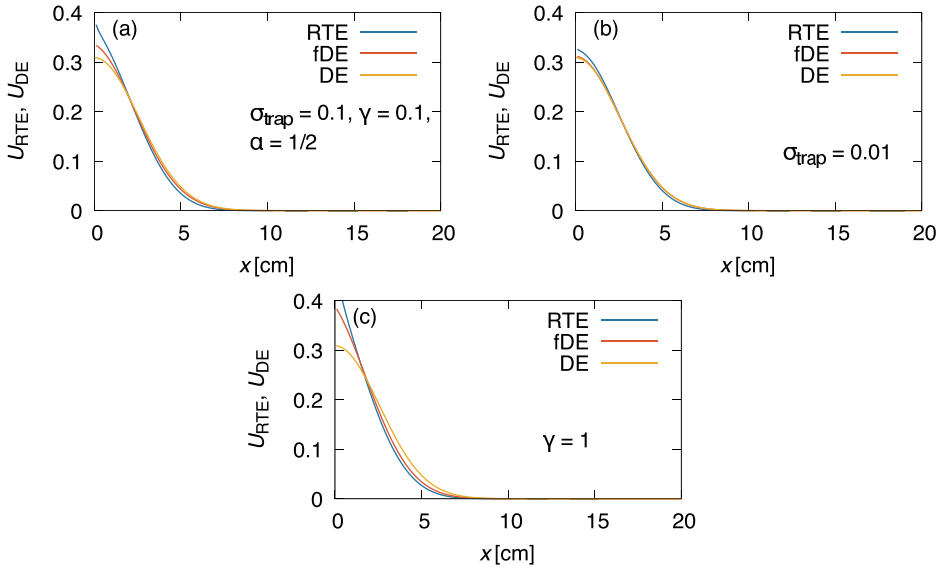


Figure 1. Comparison of the solution (blue line) to the transport equation (10), the solution (red line) to the fractional diffusion equation (13), and the solution (ocher line) to the normal diffusion (15) at $t = 10$ min. In panel (a), the parameters are set to $\sigma_{\text{trap}} = 0.1 \text{ min}^{-1}$, $\gamma = 0.1 \text{ min}$, $\alpha = 1/2$. The same parameters are used in other panels except that (b) $\sigma_{\text{trap}} = 0.01 \text{ min}^{-1}$ and (c) $\gamma = 1 \text{ min}$, respectively.

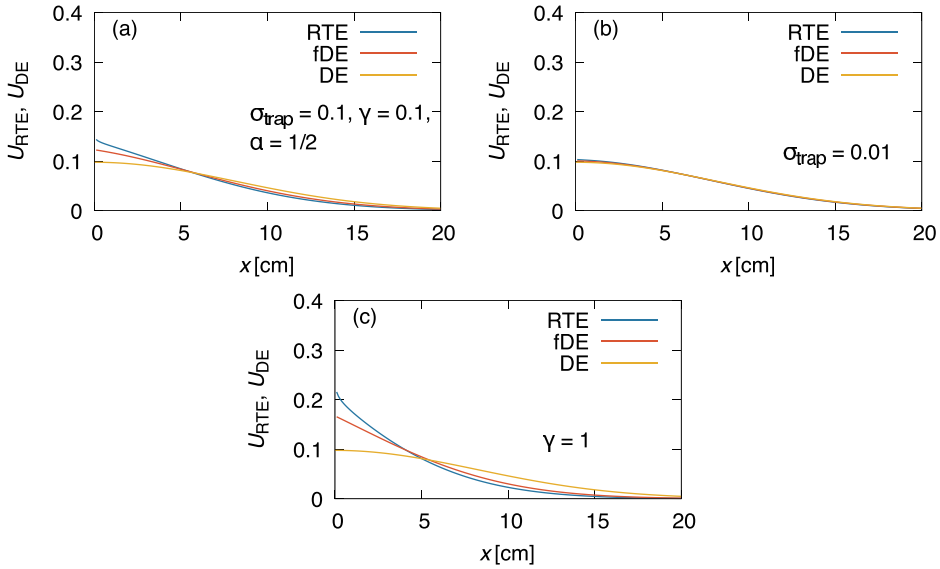


Figure 2. Same as Figure 1 but $t = 100$ min.

Figures 1 and 2 show the solution U_{RTE} to the transport equation (10) and the solution U_{DE} to the fractional diffusion equation (13) as functions of x [cm] together with the normal diffusion (15). We see that U_{RTE} and U_{DE} match as x increases. The solutions at $t = 10$ min are shown in Figure 1 and solutions at $t = 100$ min are shown in Figure 2. In panel (a), the parameters are set to $\sigma_{\text{trap}} = 0.1 \text{ min}^{-1}$, $\gamma = 0.1 \text{ min}$, $\alpha = 1/2$. In panel (b), the same parameters are used but $\sigma_{\text{trap}} = 0.01 \text{ min}^{-1}$ to show that for a small σ_{trap} , the propagation enters the diffusion regime for a shorter distance. In panel (c), $\gamma = 1 \text{ min}$ while other parameters are the same as those given in panel (a). When γ is large, U_{RTE} and U_{DE} differ more for small x . Mathematica (Wolfram Research) was used to produce Figures 1 and 2.

6. Concluding remarks

Indeed, the derived fractional equation is the time-fractional advection–dispersion equation (Schumer et al. 2003; Suzuki et al. 2016). As a related model, the tempered anomalous diffusion model is known (Meerschaert, Zhang, and Baeumer 2008). The fractional equation in this paper is derived from the radiative transport equation in a straightforward manner.

The waiting time distribution is also used for the continuous-time random walk. However, the physical meaning of the solution to (1) is clearer than random walkers. That is, ψ is the angular density of tracer particles while the relation between tracer molecules and Monte Carlo particles is vague.

If σ_{trap} is tiny (there is almost no dead-end pore), the fractional-derivative term in (9) is negligible. In this case, the usual advection–diffusion equation is recovered. Finally, we note that the diffusion approximation can similarly be done in the case of the exponential decay, $w(\tau) = \frac{1}{\tau} e^{-\tau/\gamma}$, but the resulting diffusion equation only has a first-order time-derivative term.

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References

- Adams, E. E. and Gelhar, L. W. 1992. Field study of dispersion in a heterogeneous aquifer, 2, spatial moments analysis. *Water Resour. Res.* 28 (12):3293–307. doi:[10.1029/92WR01757](https://doi.org/10.1029/92WR01757)
- Amagai, K., Hatano, Y., and Machida, M. 2021. Linear transport in porous media. *J. Comput. Theor. Transp.* 50 (5):377–89. doi:[10.1080/23324309.2020.1842453](https://doi.org/10.1080/23324309.2020.1842453)
- Amagai, K., Yamakawa, M., Machida, M., and Hatano, Y. 2020. The linear Boltzmann equation in column experiments of porous media. *Transp. Porous. Med.* 132 (2):311–31. doi:[10.1007/s11242-020-01393-1](https://doi.org/10.1007/s11242-020-01393-1)
- Bal, G. 2001. Diffusion approximation of radiative transfer equations in a channel. *Transp. Theory Stat. Phys.* 30 (2–3):269–93. doi:[10.1081/TT-100105370](https://doi.org/10.1081/TT-100105370)
- Bardos, C., Santos, R., and Sentis, R. 1984. Diffusion approximation and computation of the critical size. *Trans. Am. Math. Soc.* 284 (2):617–49. doi:[10.2307/1999099](https://doi.org/10.2307/1999099)
- Barichello, L. B. 2011. Explicit formulations for radiative transfer problems. In *Thermal measurements and inverse techniques*, ed. H. R. B. Orlande, O. Fudym, D. Mailliet, and R. M. Cotta. Boca Raton, FL: CRS Press.
- Barichello, L. B. and Siewert, C. E. 1999a. A discrete-ordinates solution for a non-grey model with complete frequency redistribution. *J. Quant. Spec. Rad. Trans.* 62 (6):665–75. doi:[10.1016/S0022-4073\(98\)00096-X](https://doi.org/10.1016/S0022-4073(98)00096-X)
- Barichello, L. B. and Siewert, C. E. 1999b. A discrete-ordinates solution for a polarization model with complete frequency redistribution. *Astro. J.* 513 (1):370–82. doi:[10.1086/306834](https://doi.org/10.1086/306834)

- Barichello, L. B., Garcia, R. D. M., and Siewert, C. E. 2000. Particular solutions for the discrete-ordinates method. *J. Quant. Spec. Rad. Trans.* 64 (3):219–26. doi:[10.1016/S0022-4073\(98\)00146-0](https://doi.org/10.1016/S0022-4073(98)00146-0)
- Barichello, L. B. and Siewert, C. E. 2002. A new version of the discrete-ordinates method. *Proceedings: International Conference on Computational Heat and Mass Transfer 2001*, Rio de Janeiro, 340–7.
- Basset, A. B. 1888a. III. On the motion of a sphere in a viscous liquid. *Philos. Trans. R. Soc. Lond. A* 179:43–63. doi:[10.1098/rsta.1888.0003](https://doi.org/10.1098/rsta.1888.0003)
- Basset, A. B. 1888b. IV. On the motion of a sphere in a viscous liquid. *Philos. Trans. R. Soc. Lond.* 43:174–5. doi:[10.1098/rspl.1887.0120](https://doi.org/10.1098/rspl.1887.0120)
- Basset, A. B. 1888c. A treatise on hydrodynamics *with numerous examples*, Vols. 1,2. Cambridge: Deighton, Bell, & Company.
- Basset, A. B. 1910. On the descent of a sphere in a viscous liquid. *Q. J. Math.* 42:369–81.
- Benson, D. A., Wheatcraft, S. W., and Meerschaert, M. M. 2000. Application of a fractional advection–dispersion equation. *Water Resour. Res.* 36 (6):1403–12. doi:[10.1029/2000WR900031](https://doi.org/10.1029/2000WR900031)
- Bensoussan, A., Lions, J.-L., and Papanicolaou, G. 1979. Boundary layers and homogenization of transport processes. *Res. Inst. Math. Sci. Kyoto Univ.* 15:53–157. doi:[10.2977/prims/1195188427](https://doi.org/10.2977/prims/1195188427)
- Bordoloi, A. D., Scheidweiler, D., Dentz, M., Bouabdellaoui, M., Abbarchi, M., and de Anna, P. 2022. Structure induced laminar vortices control anomalous dispersion in porous media. *Nat. Commun.* 13:3820. doi:[10.1038/s41467-022-31552-5](https://doi.org/10.1038/s41467-022-31552-5)
- Boussinesq, J. V. 1885. Sur la résistance qu’oppose un liquide indéfini en repos, sans pesanteur, au mouvement varié d’une sphère solide qu’il mouille sur toute sa surface, quand les vitesses restent bien continues et assez faibles pour que leurs carrés et produits soient négligeables. *Comptes Rendus Des Séances de L’Académie Des Sciences, Paris* 100:935–7.
- Caputo, M. 1967. Linear model of dissipation whose Q is almost frequency independent – II. *Geophys. J. Int.* 13 (5):529–39. doi:[10.1111/j.1365-246X.1967.tb02303.x](https://doi.org/10.1111/j.1365-246X.1967.tb02303.x)
- Chakraborty, P., Meerschaert, M. M., and Lim, C. Y. 2009. Parameter estimation for fractional transport: A particle-tracking approach. *Water Resour. Res.* 45:W10415. doi:[10.1029/2008WR007577](https://doi.org/10.1029/2008WR007577)
- Dautray, R. and Lions, J.-L. 1993. *Mathematical analysis and numerical methods for science and technology*, vol 6. Berlin: Springer Verlag.
- Davies, B. and Martin, B. 1979. Numerical inversion of the Laplace transform: A survey and comparison of methods. *J. Comput. Phys.* 33 (1):1–32. doi:[10.1016/0021-9991\(79\)90025-1](https://doi.org/10.1016/0021-9991(79)90025-1)
- Duderstadt, J. J. and Martin, W. R. 1979. *Transport theory*. New York: Wiley-Interscience Publications.
- Ganapol, B. D. 2008. *Analytical benchmarks for nuclear engineering applications case studies in neutron transport theory*. Paris: Nuclear Energy Agency, OECD.
- Hatano, Y. and Hatano, N. 1998. Dispersive transport of ions in column experiments: An explanation of long-tailed profiles. *Water Resour. Res.* 34 (5):1027–33. doi:[10.1029/98WR00214](https://doi.org/10.1029/98WR00214)
- Hou, Y., J. Jiang, and J. Wu. 2018. The form of waiting time distributions of continuous time random walk in dead-end pores. *Geofluids* 2018:1–6. doi:[10.1155/2018/8329406](https://doi.org/10.1155/2018/8329406)
- Ishimaru, A. 1978. *Wave propagation and scattering in random media*. San Diego: Academic Press.

- Karamata, J. 1931. Neuer beweis und verallgemeinerung der Tauberschen sätze, welche die Laplacesche and Stieltjessche transformation betreffen. *J. Reine Angew. Math.* 1931 (164): 27–39. doi:[10.1515/crll.1931.164.27](https://doi.org/10.1515/crll.1931.164.27)
- Kelly, J. F., Bolster, D., Meerschaert, M. M., Drummond, J. D., and Packman, A. I. 2017. FracFit: A robust parameter estimation tool for fractional calculus models. *Water Resour. Res.* 53 (3):2559–67. doi:[10.1002/2016WR019748](https://doi.org/10.1002/2016WR019748)
- Feller, W. 1971. *An introduction to probability theory and its applications*. vol. II. New York-London-Sydney: John Wiley & Sons.
- Kubica, A., Ryszewska, K., and Yamamoto, M. 2020. *Time-fractional differential equations: A theoretical introduction*. Singapore: Springer.
- Langlois, G. P., Farazmand, M., and Haller, G. 2015. Asymptotic dynamics of inertial particles with memory. *J. Nonlin. Sci.* 25:1225–55. doi:[10.1007/s00332-015-9250-0](https://doi.org/10.1007/s00332-015-9250-0)
- Larsen, E. W. and Keller, J. B. 1974. Asymptotic solution of neutron transport problems for small mean free paths. *J. Math. Phys.* 15 (1):75–81. doi:[10.1063/1.1666510](https://doi.org/10.1063/1.1666510)
- Liang, Y., Chen, W., Xu, W., and Sun, H. 2019. Distributed order Hausdorff derivative diffusion model to characterize non-Fickian diffusion in porous media. *Commun. Nonlin. Sci. Numer. Sim.* 70:384–93. doi:[10.1016/j.cnsns.2018.10.010](https://doi.org/10.1016/j.cnsns.2018.10.010)
- Mainardi, F., Mura, A., and Pagnini, G. 2010. The M -Wright function in time-fractional diffusion processes: A tutorial survey. *Int. J. Diff. Equ.* 2010 (1):104505. doi:[10.1155/2010/104505](https://doi.org/10.1155/2010/104505)
- Meerschaert, M. M., Zhang, Y., and Baeumer, B. 2008. Tempered anomalous diffusion in heterogeneous systems. *Geophys. Res. Lett.* 35 (17):L17403. doi:[10.1029/2008GL034899](https://doi.org/10.1029/2008GL034899)
- Metzler, R. and Klafter, J. 2000. The random walk's guide to anomalous diffusion: A fractional dynamics approach. *Phys. Rep.* 339 (1):1–77. doi:[10.1016/S0370-1573\(00\)00070-3](https://doi.org/10.1016/S0370-1573(00)00070-3)
- Ooura, T. and Mori, M. 1991. The double exponential formula for oscillatory functions over the half infinite interval. *J. Comp. Appl. Math.* 38 (1–3):353–60. doi:[10.1016/0377-0427\(91\)90181-I](https://doi.org/10.1016/0377-0427(91)90181-I)
- Ooura, T., and Mori, M. 1999. A robust double exponential formula for Fourier-type integrals. *J. Comp. Appl. Math.* 112 (1–2):229–41. doi:[10.1016/S0377-0427\(99\)00223-X](https://doi.org/10.1016/S0377-0427(99)00223-X)
- Podlubny, I. 1999. *Fractional differential equations*. New York: Academic Press.
- Ryzhik, L., Papanicolaou, G., and Keller, J. B. 1996. Transport equations for elastic and other waves in random media. *Wave Motion* 24 (4):327–70. doi:[10.1016/S0165-2125\(96\)00021-2](https://doi.org/10.1016/S0165-2125(96)00021-2)
- Schumer, R., Benson, D. A., Meerschaert, M. M., and Baeumer, B. 2003. Fractal mobile/immobile solute transport. *Water Resour. Res.* 39 (10):1296. doi:[10.1029/2003WR002141](https://doi.org/10.1029/2003WR002141)
- Siewert, C. E. and Wright, S. J. 1999. Efficient eigenvalue calculations in radiative transfer. *J. Quant. Spec. Rad. Trans.* 62 (6):685–8. doi:[10.1016/S0022-4073\(98\)00099-5](https://doi.org/10.1016/S0022-4073(98)00099-5)
- Sugihara, M. 1997. Optimality of the double exponential formula: Functional analysis approach. *Numer. Math.* 75:379–95. doi:[10.1007/s002110050244](https://doi.org/10.1007/s002110050244)
- Sun, H., Chen, W., and Chen, Y. 2009. Variable-order fractional differential operators in anomalous diffusion modeling. *Phys. A: Stat. Mech. Appl.* 388 (21):4586–92. doi:[10.1016/j.physa.2009.07.024](https://doi.org/10.1016/j.physa.2009.07.024)
- Sun, H., Li, Z., Zhang, Y., and Chen, W. 2017. Fractional and fractal derivative models for transient anomalous diffusion: Model comparison. *Chaos. Solitons Fract.* 102:346–53. doi:[10.1016/j.chaos.2017.03.060](https://doi.org/10.1016/j.chaos.2017.03.060)
- Suzuki, A., Hashida, T., Li, K., and Horne, R. N. 2016. Experimental tests of truncated diffusion in fault damage zones. *Water Resour. Res.* 52 (11):8578–89. doi:[10.1002/2016WR019017](https://doi.org/10.1002/2016WR019017)

- Trefethen, L. N. and Weideman, J. A. C. 2014. The exponentially convergent trapezoidal rule. *SIAM Rev.* 56 (3):385–458. doi:[10.1137/130932132](https://doi.org/10.1137/130932132)
- Wei, S., Chen, W., and Hon, Y. C. 2016. Characterizing time dependent anomalous diffusion process: A survey on fractional derivative and nonlinear models. *Phys. A: Stat. Mech. Appl.* 462:1244–51. doi:[10.1016/j.physa.2016.06.145](https://doi.org/10.1016/j.physa.2016.06.145)
- Zhang, Y., Reeves, D. M., Pohlmann, K., Chapman, J. B., and Russell, C. E. 2013. Fractional dynamics of tracer transport in fractured media from local to regional scales. *Cent. Eur. J. Phys.* 11:634–45. doi:[10.2478/s11534-013-0200-x](https://doi.org/10.2478/s11534-013-0200-x)
- Williams, M. M. R. 1992a. Stochastic problems in the transport of radioactive nuclides in fractured rock. *Nucl. Sci. Eng.* 112 (3):215–30. doi: [10.13182/NSE92-A29070](https://doi.org/10.13182/NSE92-A29070)
- Williams, M. M. R. 1992b. A new model for describing the transport of radionuclides through fractured rock. *Ann. Nucl. Energy* 19 (10-12):791–824. doi:[10.1016/0306-4549\(92\)90018-7](https://doi.org/10.1016/0306-4549(92)90018-7)
- Williams, M. M. R. 1993a. A new model for describing the transport of radionuclides through fractured rock. Part II: Numerical results. *Ann. Nucl. Energy* 20:185–202. doi:[10.1016/0306-4549\(93\)90101-t](https://doi.org/10.1016/0306-4549(93)90101-t)
- Williams, M. M. R. 1993b. Radionuclide transport in fractured rock a new model: Application and discussion. *Ann. Nucl. Energy* 20 (4):279–97. doi:[10.1016/0306-4549\(93\)90083-2](https://doi.org/10.1016/0306-4549(93)90083-2)