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The inverse Rytov series for diffuse optical tomography

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Abstract

The Rytov approximation is known in near-infrared spectroscopy including diffuse optical tomography. In diffuse optical tomography, the Rytov approximation often gives better reconstructed images than the Born approximation. Although related inverse problems are nonlinear, the Rytov approximation is almost always accompanied by the linearization of nonlinear inverse problems. In this paper, we will develop nonlinear reconstruction with the inverse Rytov series. By this, linearization is not necessary and higher order terms in the Rytov series can be used for reconstruction. The convergence and stability are discussed. We find that the inverse Rytov series has a recursive structure similar to the inverse Born series.

Keywords: Rytov series, diffuse optical tomography, inverse problems

(Some figures may appear in colour only in the online journal)

1. Introduction

We consider diffuse light propagation in a bounded domain $\Omega \subset \mathbb{R}^n$ ($n \geqslant 2$) with a smooth boundary $\partial\Omega$. In diffuse optical tomography, coefficients of the diffusion equation are determined from boundary measurements. In this paper, we consider the reconstruction of the absorption coefficient.

The time-independent diffusion equation is given by

$$\begin{cases} -D_0\Delta u + \mu_a u = f, & x \in \Omega, \\ D_0\partial_\nu u + \frac{1}{\zeta}u = 0, & x \in \partial\Omega. \end{cases} \quad (1)$$

Here, D_0, ζ are positive constants, and ∂_ν denotes the directional derivative with the outward unit vector ν normal to $\partial\Omega$. Furthermore, μ_a is the absorption coefficient and f is the source term. The outgoing light $u(x)$ is detected on a subboundary Γ of the boundary ($x \in \Gamma \subset \partial\Omega$). On the boundary, we suppose $u \in L^p(\Gamma)$ with some $p \geqslant 1$.

Since the cost function for the inverse problem of determining coefficients of the diffusion equation in (1) has a complicated landscape, the reconstructed value is trapped in a local minimum if iterative schemes such as the Levenberg–Marqusrdt, Gauss–Newton, and conjugate gradient methods are used. An alternative approach is the use of direct methods in which perturbations of coefficients are reconstructed. The Born and Rytov approximations are frequently used in cooperation with linearization of the nonlinear inverse problem. When the (first) Born approximation is compared with the (first) Rytov approximation, the superiority of the latter has been discussed [2, 12, 15].

A systematic way of inverting the Born series has been studied [18, 21, 25, 26, 28]. That is, higher-order Born approximations can be implemented with the inverse Born series. In this way, the direct methods can be applied to nonlinear inverse problems without linearization. In [17], the inverse Born series was implemented for the transport-based optical tomography. In addition to optical tomography, the Calderón problem was considered with the inverse Born series [3]. The inverse Born series was applied to inverse problems for scalar waves [13] and for electromagnetic scattering [14]. The series was developed for discrete inverse problems [8]. The technique of the inverse Born series was used to investigate the inversion of the Bremmer series [31]. The inverse Born series was extended to Banach spaces [4, 16]. Recently, a modified Born series with unconditional convergence was proposed and its inverse series was studied [1]. The convergence theorem for the inverse Born series has recently been improved [11]. See [27] for recent advances. Moreover a reduced inverse Born series was proposed [22].

Based on the success of past studies on the inverse Born series, in this paper we consider the inversion of the Rytov series. In experimental and clinical researches on optical tomography, quite often the Born approximation is impractical and tomographic images are obtained with the Rytov approximation. After linearization, the Rytov approximation was used for detecting breast cancer [6, 7] and used when the brain function was studied through the neurovascular coupling [9]. The limitation of the linear approximation has been pointed out [5].

Indeed, the inverse Rytov series was considered for the Helmholtz equation and it was numerically observed that the inverse Rytov series with the first through third approximations give better reconstructed images than the inverse Born series [32]. In [23], intermediate approximations between the Born and Rytov approximations were explored. The relation between the inverse Rytov series and Newton's method was investigated [29]. In these papers, however, no systematic way of computing higher-order terms was presented.

The remainder of the paper is organized as follows. The Born series is introduced in section 2 and the Rytov series is introduced in section 3. Then the inverse Rytov series is discussed in section 4. Section 5 is devoted to the implementation of the inverse Rytov series and numerical examples. Concluding remarks are given in section 6.

2. The Born series

Let g be a positive constant. We write

$$\mu_a(x) = g\left(1+\eta(x)\right), \quad \eta \geqslant -1.$$

We suppose that η is supported in a closed ball B_a of radius a:

$$\operatorname{supp}\eta \subset B_a \subset \Omega.$$

It will be seen below that the Born series converges for sufficiently small $a>0$. We suppose that $\eta \in L^q(B_a)$ for some $q \geqslant 2$.

Let $u_0(x)$ be the solution to the equation (1) in which $\mu_a(x)$ is replaced by g. We assume that there exists a constant $\xi > 0$ such that $\xi \leqslant u_0$ on Γ. Let $G(x,y)$ be the Green's function which corresponds to u_0. Then the following identity holds.

$$u(x) = u_0(x) - g\int_\Omega G(x,y)\eta(y)u(y)\,\mathrm{d}y.$$

From the above identity, the Born series can be constructed as

$$u = u_0 + u_1 + \ldots,$$

where

$$u_j(x) = -g\int_\Omega G(x,y)\eta(y)u_{j-1}(y)\,\mathrm{d}y \quad (j=1,2,\ldots).$$

The first two terms of the Born series are obtained as

$$u_1(x) = -g\int_\Omega G(x,y)\eta(y)u_0(y)\,\mathrm{d}y,$$
$$u_2(x) = g^2\int_\Omega\int_\Omega G(x,y)\eta(y)G(y,z)\eta(z)u_0(z)\,\mathrm{d}y\mathrm{d}z.$$

Let us introduce the multilinear operators $K_j : L^q(B_a)\times\cdots\times L^q(B_a) \to L^p(\Gamma)$ such that

$$u_j = -K_j\eta^{\otimes j},$$

where $\eta^{\otimes j} = \eta\otimes\cdots\otimes\eta$ is the j-fold tensor product. Here we have

$$K_1\eta = g\int_{B_a} G(x,y)u_0(y)\eta(y)\,\mathrm{d}y,$$
$$K_2\eta\otimes\eta = -g^2\int_{B_a}\int_{B_a} G(x,y)G(y,z)u_0(z)\eta(y)\eta(z)\,\mathrm{d}y\mathrm{d}z.$$

In general, the jth term is given by

$$K_j\eta^{\otimes j} = (-1)^{j+1}g^j\int_{B_a\times\cdots\times B_a} G(x,y_1)G(y_1,y_2)\cdots G(y_{j-1},y_j)$$
$$\times\, u_0(y_j)\eta(y_1)\cdots\eta(y_j)\,\mathrm{d}y_1\cdots\mathrm{d}y_j.$$

Let us define the operators $\check{K}_j : L^q(B_a)\times\cdots\times L^q(B_a) \to L^p(\Gamma)$ such that

$$\frac{1}{u_0}K_j\eta^{\otimes j} = \check{K}_j\eta^{\otimes j}.$$

We introduce

$$\mu = g\sup_{x\in B_a}\|G(x,\cdot)\|_{L^r(B_a)}, \quad \nu = g|B_a|^{1/r}\sup_{y_1,y_2\in B_a}\left\|G(\cdot,y_1)\frac{u_0(y_2)}{u_0(\cdot)}\right\|_{L^p(\Gamma)},$$

where $r = q/(q-1)$.

Lemma 2.1. *For* $j = 1, 2, \ldots$, $\|\check{K}_j\| \leqslant \nu\mu^{j-1}$.

Proof. For any $f_i \in L^q(B_a)$ $(i = 1, \ldots, j)$, the multilinear operators $\check{K}_j$ are written as

$$(\check{K}_j f_1 \otimes \cdots \otimes f_j)(x) = \frac{(-1)^{j+1} g^j}{u_0(x)} \int_{B_a \times \cdots \times B_a} G(x, y_1) G(y_1, y_2) \cdots G(y_{j-1}, y_j) \times u_0(y_j) f_1(y_1) \cdots f_j(y_j) \, \mathrm{d}y_1 \cdots \mathrm{d}y_j, \quad x \in \Gamma.$$

Using Hölder's inequality, we have

$$\begin{aligned}
&\|\check{K}_j f_1 \otimes \cdots \otimes f_j\|^p_{L^p(\Gamma)} \\
&= (g^j)^p \int_\Gamma \left| \int_{B_a \times \cdots \times B_a} G(x, y_1) G(y_1, y_2) \cdots G(y_{j-1}, y_j) \right. \\
&\quad \left. \times \frac{u_0(y_j)}{u_0(x)} f_1(y_1) \cdots f_j(y_j) \, \mathrm{d}y_1 \cdots \mathrm{d}y_j \right|^p \mathrm{d}x \\
&\leqslant g^{jp} \int_\Gamma \left| \left(\int_{B_a \times \cdots \times B_a} |f_1(y_1) \cdots f_j(y_j)|^q \, \mathrm{d}y_1 \cdots \mathrm{d}y_j \right)^{1/q} \right. \\
&\quad \left. \times \left(\int_{B_a \times \cdots \times B_a} \left| G(x, y_1) G(y_1, y_2) \cdots G(y_{j-1}, y_j) \frac{u_0(y_j)}{u_0(x)} \right|^r \mathrm{d}y_1 \cdots \mathrm{d}y_j \right)^{1/r} \right|^p \mathrm{d}x \\
&\leqslant g^{jp} \|f_1\|^p_{L^q(B_a)} \cdots \|f_j\|^p_{L^q(B_a)} \int_\Gamma \left| \sup_{y_1, y_j \in B_a} G(x, y_1) \frac{u_0(y_j)}{u_0(x)} \right|^p \mathrm{d}x \\
&\quad \times \left(\int_{B_a \times \cdots \times B_a} |G(y_1, y_2) \cdots G(y_{j-1}, y_j)|^r \, \mathrm{d}y_1 \cdots \mathrm{d}y_j \right)^{p/r}.
\end{aligned}$$

We define

$$I_{j-1} = g^{j-1} \left(\int_{B_a \times \cdots \times B_a} |G(y_1, y_2) \cdots G(y_{j-1}, y_j)|^r \, \mathrm{d}y_1 \cdots \mathrm{d}y_j \right)^{1/r}.$$

Similar to the calculation in [25], we have

$$I_{j-1} \leqslant \mu I_{j-2}, \quad I_1 \leqslant |B_a|^{1/r} \mu.$$

Hence,

$$I_{j-1} \leqslant \mu^{j-1} |B_a|^{1/r} \quad (j = 2, 3, \ldots).$$

We obtain

$$\|\check{K}_j f_1 \otimes \cdots \otimes f_j\|^p_{L^p(\Gamma)} \leqslant \|f_1\|^p_{L^q(B_a)} \cdots \|f_j\|^p_{L^q(B_a)} \nu^p \mu^{p(j-1)}.$$

Therefore,

$$\|\check{K}_j\| = \sup_{\substack{f_1, \ldots, f_j \in L^q(B_a) \\ f_i \neq 0 \ (i = 1, \ldots, j)}} \frac{\left\|\check{K}_j f_1 \otimes \cdots \otimes f_j\right\|_{L^p(\Gamma)}}{\|f_1\|_{L^q(B_a)} \cdots \|f_j\|_{L^q(B_a)}} \leqslant \nu\mu^{j-1}.$$

□

3. The Rytov series

Let us consider the Rytov series: $u = u_0 e^{-\psi_1 - \psi_2 - \cdots}$. The function ψ_j $(j = 1, 2, \ldots)$ is proportional to g^j. In particular, we consider boundary values of u, u_0 at $x \in \Gamma$. We introduce

$$\psi = \psi(x) = \ln \frac{u_0(x)}{u(x)}, \quad x \in \Gamma.$$

We assume $\psi \in L^p(\Gamma)$. We have

$$\begin{aligned}
-\psi &= \ln \frac{u_0 + u_1 + \cdots}{u_0} = \ln \left(1 + \sum_{j=1}^{\infty} \frac{u_j}{u_0} \right) \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left(\sum_{j=1}^{\infty} \frac{u_j}{u_0} \right)^k \\
&= \frac{u_1 + u_2 + \cdots}{u_0} - \frac{(u_1 + u_2 + \cdots)^2}{2u_0^2} + \frac{(u_1 + u_2 + \cdots)^3}{3u_0^3} - \cdots \\
&= -\psi_1 - \psi_2 - \cdots .
\end{aligned}$$

By collecting the first- and second-order terms, the first two terms of the Rytov series are explicitly written as

$$\psi_1 = -\frac{u_1}{u_0}, \quad \psi_2 = -\frac{u_2}{u_0} + \frac{1}{2} \left(\frac{u_1}{u_0} \right)^2 .$$

In general, we have

$$\psi_j = \sum_{m=1}^{j} \frac{(-1)^m}{m u_0^m} \sum_{i_1 + \cdots + i_m = j} u_{i_1} \cdots u_{i_m}, \quad j = 1, 2, \ldots .$$

We note that the number of jth order terms in $(u_1 + \cdots)^m$ is

$$\binom{j-1}{m-1}.$$

In total, the number of terms in ψ_j is

$$\sum_{m=1}^{j-1} \binom{j-1}{m-1} = 2^{j-1}.$$

We introduce the forward operators $J_j : L^q(B_a) \times \cdots \times L^q(B_a) \to L^p(\Gamma)$ such that

$$\psi_j = J_j \eta^{\otimes j} \quad (j = 1, 2, \ldots).$$

Note that J_j are multilinear. We have

$$\begin{aligned}
J_1 \eta &= \check{K}_1 \eta = \frac{g}{u_0(x)} \int_\Omega G(x,y) u_0(y) \eta(y) \, \mathrm{d}y, \\
J_2 \eta \otimes \eta &= \check{K}_2 \eta \otimes \eta + \frac{1}{2} \left(\check{K}_1 \eta \right)^2 \\
&= \frac{g^2}{u_0(x)} \int_\Omega \int_\Omega G(x,y) G(y,z) u_0(z) \eta(y) \eta(z) \, \mathrm{d}y \mathrm{d}z + \frac{g^2}{2u_0(x)^2} \left(\int_\Omega G(x,y) u_0(y) \eta(y) \, \mathrm{d}y \right)^2 .
\end{aligned}$$

In general, the jth term is given by

$$J_j\eta^{\otimes j} = \sum_{m=1}^{j} \frac{1}{m} \sum_{i_1+\cdots+i_m=j} \left(\check{K}_{i_1}\eta^{\otimes i_1}\right)\cdots\left(\check{K}_{i_m}\eta^{\otimes i_m}\right).$$

Lemma 3.1. *We have* $\|J_j\| \leqslant \nu\,(\mu+\nu)^{j-1}$ *for* $j = 1, 2, \ldots$. *Moreover the Rytov series converges if* $\|\eta\|_{L^q(B_a)} < (\mu+\nu)^{-1}$.

Proof. We note the binomial formula:

$$x(x+y)^{j-1} = \sum_{m=1}^{j} \begin{pmatrix} j-1 \\ m-1 \end{pmatrix} x^m y^{j-m}. \tag{2}$$

We have

$$\begin{aligned}
\|J_j\| &\leqslant \sum_{m=1}^{j} \frac{1}{m} \sum_{i_1+\cdots+i_m=j} \|\check{K}_{i_1}\|\cdots\|\check{K}_{i_m}\| \\
&\leqslant \sum_{m=1}^{j} \begin{pmatrix} j-1 \\ m-1 \end{pmatrix} \nu^m \mu^{j-m} \\
&= \nu\,(\mu+\nu)^{j-1}.
\end{aligned}$$

Since we have

$$\begin{aligned}
\sum_{j=1}^{\infty} \|\psi_j\|_{L^p(\Gamma)} = \sum_{j=1}^{\infty} \|J_j\eta\otimes\cdots\otimes\eta\|_{L^p(\Gamma)} &\leqslant \sum_{j=1}^{\infty} \|J_j\|\|\eta\|^j_{L^q(B_a)} \\
&\leqslant \nu\,(\mu+\nu)^{-1} \sum_{j=1}^{\infty} (\mu+\nu)^j \|\eta\|^j_{L^q(B_a)},
\end{aligned}$$

the series converges if $\|\eta\|_{L^q(B_a)} < (\mu+\nu)^{-1}$.

□

4. Inverse Rytov series

We begin by formally expanding the perturbation η as

$$\begin{aligned}
\eta &= \eta_1 + \eta_2 + \cdots \\
&= \mathcal{J}_1\psi + \mathcal{J}_2\psi\otimes\psi + \cdots.
\end{aligned}$$

We refer to the above series as the inverse Rytov series. If we substitute the series $\psi = J_1\eta + J_2\eta\otimes\eta + \cdots$, we have

$$\begin{aligned}
\eta &= \mathcal{J}_1\left(J_1\eta + J_2\eta\otimes\eta + \cdots\right) + \mathcal{J}_2\left(J_1\eta + J_2\eta\otimes\eta + \cdots\right)\otimes\left(J_1\eta + J_2\eta\otimes\eta + \cdots\right) + \cdots \\
&= \mathcal{J}_1 J_1\eta + \left(\mathcal{J}_1 J_2 + \mathcal{J}_2 J_1\otimes J_1\right)\eta\otimes\eta + \cdots.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
&\mathcal{J}_1 J_2 + \mathcal{J}_2 J_1\otimes J_1 = 0, \\
&\mathcal{J}_3 J_1\otimes J_1\otimes J_1 + \mathcal{J}_2 J_1\otimes J_2 + \mathcal{J}_2 J_2\otimes J_1 + \mathcal{J}_1 J_3 = 0, \ldots.
\end{aligned}$$

Indeed, the equality $\eta = \mathcal{J}_1 J_1 \eta$ does not hold due to the ill-posedness of this inverse problem. To consider $\mathcal{J}_1$, let us introduce η^* as [17]

$$\eta^* = \operatorname*{arg\,min}_{\eta \in B_a} \left(\frac{1}{2} \|J_1 \eta - \psi\|^2_{L^p(\Gamma)} + \alpha R(\eta) \right),$$

where $R(\eta)$ is a penalty function with a regularization parameter $\alpha > 0$ [10, 24, 30]. The regularized pseudoinverse of J_1 is defined as $\mathcal{J}_1 : \psi \mapsto \eta^*$. With this operator $\mathcal{J}_1$, we have

$$\mathcal{J}_2 \psi \otimes \psi = -\mathcal{J}_1 J_2 (\mathcal{J}_1 \otimes \mathcal{J}_1)(\psi \otimes \psi) = -\mathcal{J}_1 \left[\check{K}_2 (\mathcal{J}_1 \otimes \mathcal{J}_1)(\psi \otimes \psi) + \frac{1}{2} \left(\check{K}_1 \mathcal{J}_1 \psi \right)^2 \right],$$

and

$$\begin{aligned} \mathcal{J}_3 \psi \otimes \psi \otimes \psi &= -\left(\mathcal{J}_2 J_1 \otimes J_2 + \mathcal{J}_2 J_2 \otimes J_1 + \mathcal{J}_1 J_3 \right) (\mathcal{J}_1 \otimes \mathcal{J}_1 \otimes \mathcal{J}_1)(\psi \otimes \psi \otimes \psi) \\ &= -\mathcal{J}_2 (J_1 \mathcal{J}_1 \psi) \otimes \left[\check{K}_2 \mathcal{J}_1 \psi \otimes \mathcal{J}_1 \psi + \frac{1}{2} (\check{K}_1 \mathcal{J}_1 \psi)^2 \right] \\ &\quad - \mathcal{J}_2 \left[\check{K}_2 \mathcal{J}_1 \psi \otimes \mathcal{J}_1 \psi + \frac{1}{2} (\check{K}_1 \mathcal{J}_1 \psi)^2 \right] \otimes J_1 \mathcal{J}_1 \psi \\ &\quad - \mathcal{J}_1 \left[\check{K}_3 (\mathcal{J}_1 \psi)^{\otimes 3} + (\check{K}_1 \mathcal{J}_1 \psi)(\check{K}_2 \mathcal{J}_1 \psi \otimes \mathcal{J}_1 \psi) + \frac{1}{3} (\check{K}_1 \mathcal{J}_1 \psi)^3 \right]. \end{aligned}$$

For $j \geqslant 2$, we have

$$\mathcal{J}_j = -\left(\sum_{m=1}^{j-1} \mathcal{J}_m \sum_{i_1 + \cdots + i_m = j} J_{i_1} \otimes \cdots \otimes J_{i_m} \right) \mathcal{J}_1 \otimes \cdots \otimes \mathcal{J}_1.$$

Theorem 4.1. *Assume that there exists a constant $M_1 < 1$ such that $(\mu + 2\nu)\|\mathcal{J}_1\| \leqslant M_1$. Then the operator $\mathcal{J}_j : L^p(\Gamma) \times \cdots \times L^p(\Gamma) \to L^q(B_a)$ is bounded and*

$$\|\mathcal{J}_j\| \leqslant C_1 (\mu + 2\nu)^j \|\mathcal{J}_1\|,$$

where constant $C_1 = C_1(M_1) > 0$ is independent of j. Moreover for any $\psi \in L^p(\Gamma)$, there exists $C_2 = C_2(M_1, \mu, \nu)$ such that

$$\left\| \mathcal{J}_j \psi^{\otimes j} \right\|_{L^q(B_a)} \leqslant C_2 (\mu + 2\nu)^j \|\mathcal{J}_1 \psi\|^j_{L^q(B_a)}.$$

Proof. We find that for $j \geqslant 2$,

$$\begin{aligned} \|\mathcal{J}_j\| &= \left\| \left(\sum_{m=1}^{j-1} \mathcal{J}_m \sum_{i_1 + \cdots + i_m = j} J_{i_1} \otimes \cdots \otimes J_{i_m} \right) \mathcal{J}_1 \otimes \cdots \otimes \mathcal{J}_1 \right\| \\ &\leqslant \left\| \sum_{m=1}^{j-1} \mathcal{J}_m \nu^m \sum_{i_1 + \cdots + i_m = j} (\mu + \nu)^{i_1 - 1} \cdots (\mu + \nu)^{i_m - 1} \right\| \|\mathcal{J}_1\|^j \\ &\leqslant \sum_{m=1}^{j-1} \|\mathcal{J}_m\| \nu^m \begin{pmatrix} j-1 \\ m-1 \end{pmatrix} (\mu + \nu)^{j-m} \|\mathcal{J}_1\|^j \\ &\leqslant \|\mathcal{J}_1\|^j \left(\sum_{m=1}^{j-1} \|\mathcal{J}_m\| \right) \left(\sum_{m=1}^{j-1} \begin{pmatrix} j-1 \\ m-1 \end{pmatrix} \nu^m (\mu + \nu)^{j-m} \right). \end{aligned}$$

By using (2), we have

$$\begin{aligned}\|\mathcal{J}_j\| &\leqslant \|\mathcal{J}_1\|^j \left(\sum_{m=1}^{j-1}\|\mathcal{J}_m\|\right)\left(\nu\,(\mu+2\nu)^{j-1}-\nu^j\right)\\ &\leqslant \nu\|\mathcal{J}_1\|^j\,(\mu+2\nu)^{j-1}\sum_{m=1}^{j-1}\|\mathcal{J}_m\|\\ &\leqslant \|\mathcal{J}_1\|^j\,(\mu+2\nu)^j\sum_{m=1}^{j-1}\|\mathcal{J}_m\|.\end{aligned}$$

By noticing the recursive structure of the above inequality, we can write

$$\|\mathcal{J}_j\| \leqslant c_j\left[(\mu+2\nu)\,\|\mathcal{J}_1\|\right]^j\|\mathcal{J}_1\|,$$

where

$$c_{j+1} = c_j + \left[(\mu+2\nu)\,\|\mathcal{J}_1\|\right]^j c_j, \quad c_2 = 1.$$

Hence we obtain

$$c_j = \prod_{m=2}^{j-1}\left(1+\left[(\mu+2\nu)\,\|\mathcal{J}_1\|\right]^m\right), \quad j\geqslant 3.$$

We note that

$$\begin{aligned}\ln c_j &\leqslant \sum_{m=1}^{j-1}\ln\left(1+\left[(\mu+2\nu)\,\|\mathcal{J}_1\|\right]^m\right)\\ &\leqslant \sum_{m=1}^{j-1}\left[(\mu+2\nu)\,\|\mathcal{J}_1\|\right]^m\\ &\leqslant \frac{1}{1-(\mu+2\nu)\,\|\mathcal{J}_1\|}\\ &\leqslant \frac{1}{1-M_1}.\end{aligned}$$

Thus c_j $(j\geqslant 2)$ are bounded. We put $C_1 = \exp(1/(1-M_1))$.

We note that

$$\left\|\mathcal{J}_j\psi^{\otimes j}\right\|_{L^q(B_a)} \leqslant \|\mathcal{J}_1\psi\|^j_{L^q(B_a)}\,(\mu+2\nu)^j\sum_{m=1}^{j-1}\|\mathcal{J}_m\|,$$

and

$$\sum_{m=1}^{j-1}\|\mathcal{J}_m\| \leqslant C_1\|\mathcal{J}_1\|\,(\mu+2\nu)\,\frac{1-(\mu+2\nu)^{j-1}}{1-(\mu+2\nu)}.$$

Hence we obtain

$$\begin{aligned}\left\|\mathcal{J}_j\psi^{\otimes j}\right\|_{L^q(B_a)} &\leqslant C_1\,(\mu+2\nu)^{j+1}\,\|\mathcal{J}_1\|\,\frac{1-(\mu+2\nu)^{j-1}}{1-(\mu+2\nu)}\,\|\mathcal{J}_1\psi\|^j_{L^q(B_a)}\\ &\leqslant \frac{C_1M_1}{1-(\mu+2\nu)}\,(\mu+2\nu)^j\,\|\mathcal{J}_1\psi\|^j_{L^q(B_a)}.\end{aligned}$$

The proof is complete if we set

$$C_2 = \frac{C_1 M_1}{1 - (\mu + 2\nu)}.$$

□

Let us consider the convergence of the inverse Rytov series. If the inverse Rytov series converges, we write

$$\eta \approx \widetilde{\eta},$$

where

$$\widetilde{\eta} = \sum_{j=1}^{\infty} \mathcal{J}_j \psi^{\otimes j}.$$

Theorem 4.2. *Assume that there exists a constant* $M_1 < 1$ *such that* $(\mu + 2\nu)\|\mathcal{J}_1\| \leqslant M_1$. *Suppose that* $\|\mathcal{J}_1\psi\|_{L^q(B_a)} < (\mu + 2\nu)^{-1}$. *Let* $M_2 = \max(\|\eta\|_{L^q(B_a)}, \|\mathcal{J}_1 J_1 \eta\|_{L^q(B_a)})$. *We assume that* $M_2 < (\mu + 2\nu)^{-1}$. *Then for any* $N \in \mathbb{N}$ *there exists constants* $C_3 = C_3(M_1, M_2, \mu, \nu) > 0$ *such that*

$$\left\| \eta - \sum_{j=1}^{N} \mathcal{J}_j \psi^{\otimes j} \right\|_{L^q(B_a)} \leqslant C_3 \|(I - \mathcal{J}_1 J_1)\eta\|_{L^q(B_a)} + C_2 \frac{\left[(\mu + 2\nu)\|\mathcal{J}_1\psi\|_{L^q(B_a)}\right]^{N+1}}{1 - (\mu + 2\nu)\|\mathcal{J}_1\psi\|_{L^q(B_a)}},$$

where constant $C_2 > 0$ *is given in theorem* 4.1.

Proof. If we expand ψ in the inverse Rytov series by the Rytov series, we can write

$$\widetilde{\eta} = \sum_{j=1}^{\infty} \widetilde{\mathcal{J}}_j \eta \otimes \cdots \otimes \eta,$$

where

$$\widetilde{\mathcal{J}}_1 = \mathcal{J}_1 J_1,$$

and

$$\widetilde{\mathcal{J}}_j = \left(\sum_{m=1}^{j-1} \mathcal{J}_m \sum_{i_1 + \cdots + i_m = j} J_{i_1} \otimes \cdots \otimes J_{i_m} \right) + \mathcal{J}_j J_1 \otimes \cdots \otimes J_1, \quad j \geqslant 2.$$

We have

$$\widetilde{\mathcal{J}}_j = \sum_{m=1}^{j-1} \mathcal{J}_m \sum_{i_1 + \cdots + i_m = j} J_{i_1} \otimes \cdots \otimes J_{i_m} \left(I - \mathcal{J}_1 J_1 \otimes \cdots \otimes \mathcal{J}_1 J_1 \right).$$

Since

$$\eta - \widetilde{\eta} = (I - \mathcal{J}_1 J_1)\eta - \mathcal{J}_1 J_2 \left(\eta \otimes \eta - \mathcal{J}_1 J_1 \eta \otimes \mathcal{J}_1 J_1 \eta \right) + \cdots,$$

we have

$$\begin{aligned} \|\eta - \widetilde{\eta}\|_{L^q(B_a)} \leqslant & \sum_{j=1}^{\infty} \sum_{m=1}^{j-1} \sum_{i_1 + \cdots + i_m = j} \|\mathcal{J}_m\| \|\mathcal{J}_{i_1}\| \cdots \|\mathcal{J}_{i_m}\| \\ & \times \|(\eta \otimes \cdots \otimes \eta) - (\mathcal{J}_1 J_1 \eta \otimes \cdots \otimes \mathcal{J}_1 J_1 \eta)\|_{L^q(B_a^j)}. \end{aligned}$$

We note the identity

$$
\begin{aligned}
&(\eta_1 \otimes \cdots \otimes \eta_1) - (\eta_2 \otimes \cdots \otimes \eta_2) \\
&\quad = (\eta_1 - \eta_2) \otimes \eta_2 \otimes \cdots \otimes \eta_2 + \eta_1 \otimes (\eta_1 - \eta_2) \otimes \eta_2 \otimes \cdots \otimes \eta_2 + \cdots \\
&\qquad + \eta_1 \otimes \eta_1 \otimes \cdots \otimes (\eta_1 - \eta_2) \otimes \eta_2 + \eta_1 \otimes \eta_1 \otimes \cdots \otimes \eta_1 \otimes (\eta_1 - \eta_2).
\end{aligned}
$$

Hence,

$$
\|\eta \otimes \cdots \otimes \eta - \mathcal{J}_1 J_1 \eta \otimes \cdots \otimes \mathcal{J}_1 J_1 \eta\|_{L^q(B_a^j)} \leqslant j M_2^{j-1} \|\eta - \mathcal{J}_1 J_1 \eta\|_{L^q(B_a)}.
$$

We obtain

$$
\|\eta - \widetilde{\eta}\|_{L^q(B_a)} \leqslant \sum_{j=1}^{\infty} \sum_{m=1}^{j-1} \sum_{i_1+\cdots+i_m=j} \|\mathcal{J}_m\| \|\mathcal{J}_{i_1}\| \cdots \|\mathcal{J}_{i_m}\| j M_2^{j-1} \|\eta - \mathcal{J}_1 J_1 \eta\|_{L^q(B_a)}.
$$

Furthermore,

$$
\begin{aligned}
&\|\eta - \widetilde{\eta}\|_{L^q(B_a)} \\
&\quad \leqslant \sum_{j=1}^{\infty} \sum_{m=1}^{j-1} j M_2^{j-1} \|\mathcal{J}_m\| \begin{pmatrix} j-1 \\ m-1 \end{pmatrix} \nu^m (\mu+\nu)^{j-m} \|\eta - \mathcal{J}_1 J_1 \eta\|_{L^q(B_a)} \\
&\quad \leqslant \|\eta - \mathcal{J}_1 J_1 \eta\|_{L^q(B_a)} \sum_{j=1}^{\infty} j M_2^{j-1} \left(\sum_{m=1}^{j-1} \|\mathcal{J}_m\| \right) \left(\sum_{m=1}^{j-1} \begin{pmatrix} j-1 \\ m-1 \end{pmatrix} \nu^m (\mu+\nu)^{j-m} \right) \\
&\quad = \nu \|\eta - \mathcal{J}_1 J_1 \eta\|_{L^q(B_a)} \sum_{j=1}^{\infty} j M_2^{j-1} \left(\sum_{m=1}^{j-1} \|\mathcal{J}_m\| \right) \left[(\mu+2\nu)^{j-1} - \nu^{j-1} \right] \\
&\quad \leqslant \|\eta - \mathcal{J}_1 J_1 \eta\|_{L^q(B_a)} \sum_{j=1}^{\infty} j M_2^{j-1} (\mu+2\nu)^j \left(\sum_{m=1}^{j-1} \|\mathcal{J}_m\| \right).
\end{aligned}
$$

Using $C_1 > 0$ in theorem 4.1, we obtain

$$
\begin{aligned}
\|\eta - \widetilde{\eta}\|_{L^q(B_a)} &\leqslant C_1 \|\mathcal{J}_1\| \|\eta - \mathcal{J}_1 J_1 \eta\|_{L^q(B_a)} \sum_{j=1}^{\infty} j M_2^{j-1} (\mu+2\nu)^{j+1} \frac{1-(\mu+2\nu)^{j-1}}{1-(\mu+2\nu)} \\
&\leqslant C_1 \|\eta - \mathcal{J}_1 J_1 \eta\|_{L^q(B_a)} \frac{\mu+2\nu}{1-(\mu+2\nu)} \sum_{j=1}^{\infty} j \left[M_2 (\mu+2\nu)^{j-1} \right] \\
&= C_3 \|\eta - \mathcal{J}_1 J_1 \eta\|_{L^q(B_a)},
\end{aligned}
$$

where

$$
C_3 = C_1 \frac{\mu+2\nu}{1-(\mu+2\nu)} \sum_{j=1}^{\infty} j \left[M_2 (\mu+2\nu)^{j-1} \right].
$$

We have

$$
\|\widetilde{\eta}\|_{L^q(B_a)} \leqslant \sum_{j=1}^{\infty} \|\mathcal{J}_j \psi^{\otimes j}\|_{L^q(B_a)} \leqslant C_2 \sum_{j=1}^{\infty} (\mu+2\nu)^j \|\mathcal{J}_1 \psi\|_{L^q(B_a)}^j.
$$

Hence $\widetilde{\eta}$ converges. We note that

$$\begin{aligned}\left\|\widetilde{\eta}-\sum_{j=1}^{N}\mathcal{J}_j\psi^{\otimes j}\right\|_{L^q(B_a)} &\leqslant \sum_{j=N+1}^{\infty}\|\mathcal{J}_j\psi\otimes\cdots\otimes\psi\|_{L^q(B_a)}\\ &\leqslant C_2\sum_{j=Ns+1}^{\infty}(\mu+2\nu)^j\|\mathcal{J}_1\psi\|_{L^q(B_a)}^j\\ &= C_2\frac{\left[(\mu+2\nu)\|\mathcal{J}_1\psi\|_{L^q(B_a)}\right]^{N+1}}{1-(\mu+2\nu)\|\mathcal{J}_1\psi\|_{L^q(B_a)}}\end{aligned}$$

The proof is complete. □

The stability of the reconstruction is studied as follows.

Theorem 4.3. *Assume that there exists a constant $M_1<1$ such that $(\mu+2\nu)\|\mathcal{J}_1\|\leqslant M_1$. Let η_1,η_2 denote the limits of the inverse Rytov series corresponding to some ψ_1,ψ_2. We suppose that $M_1M_3<1$, where $M_3=\max(\|\psi_1\|_{L^p(\Gamma)},\|\psi_2\|_{L^p(\Gamma)})$. Then there exists $C_4=C_4(M_1,M_3,\mu,\nu)>0$ such that*

$$\|\eta_1-\eta_2\|_{L^q(B_a)}<C_4\|\psi_1-\psi_2\|_{L^p(\Gamma)}.$$

Proof. We begin with the following inequality.

$$\|\eta_1-\eta_2\|_{L^q(B_a)}\leqslant\sum_{j=1}^{\infty}\|\mathcal{J}_j\psi_1\otimes\cdots\otimes\psi_1-\mathcal{J}_j\psi_2\otimes\cdots\otimes\psi_2\|_{L^q(B_a)}.$$

We note that

$$\begin{aligned}&(\psi_1\otimes\cdots\otimes\psi_1)-(\psi_2\otimes\cdots\otimes\psi_2)\\ &\quad=(\psi_1-\psi_2)\otimes\psi_2\otimes\cdots\otimes\psi_2+\psi_1\otimes(\psi_1-\psi_2)\otimes\psi_2\otimes\cdots\otimes\psi_2+\cdots\\ &\qquad+\psi_1\otimes\cdots\otimes\psi_1\otimes(\psi_1-\psi_2)\otimes\psi_2+\psi_1\otimes\cdots\otimes\psi_1\otimes(\psi_1-\psi_2).\end{aligned}$$

We obtain

$$\begin{aligned}&\|\eta_1-\eta_2\|_{L^q(B_a)}\\ &\quad\leqslant\sum_{j=1}^{\infty}\|\mathcal{J}_j\|\sum_{k=1}^{j}\|\psi_1\|_{L^p(\Gamma)}\cdots\|\psi_1\|_{L^p(\Gamma)}\|(\psi_1-\psi_2)\|_{L^p(\Gamma)}\|\psi_2\|_{L^p(\Gamma)}\cdots\|\psi_2\|_{L^p(\Gamma)},\end{aligned}$$

where $\|(\psi_1-\psi_2)\|_{L^p(\Gamma)}$ is in the kth position of the product. Furthermore,

$$\begin{aligned}\|\eta_1-\eta_2\|_{L^q(B_a)}&\leqslant\sum_{j=1}^{\infty}j\|\mathcal{J}_j\|M_3^{j-1}\|\psi_1-\psi_2\|_{L^p(\Gamma)}\\ &\leqslant C_1\|\mathcal{J}_1\|\|\psi_1-\psi_2\|_{L^p(\Gamma)}\sum_{j=1}^{\infty}j(\mu+2\nu)^jM_3^{j-1}\\ &\leqslant\frac{C_1}{M_3}\|\psi_1-\psi_2\|_{L^p(\Gamma)}\sum_{j=1}^{\infty}j(\mu+2\nu)^{j-1}M_3^{j-1}.\end{aligned}$$

The proof is complete if we put

$$C_4 = C_1 \sum_{j=1}^{\infty} j\,(\mu + 2\nu)^{j-1} M_3^{j-2}.$$

□

5. Two-dimensional radial problem

5.1. Setup

Let us assume the two-dimensional radial geometry, which was considered in [26]. We consider diffuse optical tomography for this domain. In the polar coordinate system we have $x = (r,\theta)$, where r is the radial coordinate and θ is the angular coordinate. Let Ω be the disk of radius R centered at the origin. Assuming that η has the radial symmetry, we write

$$\eta(x) = \eta(r), \quad 0 < r < R.$$

Let us suppose that η is given by

$$\eta(r) = \begin{cases} \eta_a, & 0 \leqslant r \leqslant R_a, \\ 0, & R_a < r \leqslant R. \end{cases}$$

Although point sources were used in [26], here we assume the following spatially oscillating source term for diffuse optical tomography in spatial frequency domain.

$$f(r,\theta) = e^{i\alpha\theta}\frac{1}{r}\delta(r-R), \quad \alpha = 1,\ldots,M_S.$$

Hereafter we write

$$g = k^2, \quad k > 0.$$

We define $\ell = \zeta D_0$ and set $D_0 = 1$. We write $\Omega_1 = \{x;\ |x| \leqslant R_a\}$, $\Omega_2 = \{x;\ R_a < |x| < R\}$, and $k_a = k\sqrt{1+\eta_a}$. Let r_x, r_y be the radial coordinates of x, y. Let θ_x, θ_y be the angular coordinates of x, y.

5.2. Forward problem

Let us express the Green's function $G(x,y)$, which has the source term $\frac{1}{r_x}\delta(r_x - r_y)\delta(\theta_x - \theta_y)$, as [19]

$$G(x,y) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{in(\theta_x - \theta_y)} g_n(r_x, r_y),$$

where $g_n(r,r')$ satisfies

$$r^2\partial_r^2 g_n(r,r') + r\partial_r g_n(r,r') - \left(k^2r^2 + n^2\right) g_n(r,r') = -r\delta(r-r'),$$
$$g_n(R,r') + \ell\partial_r g_n(R,r') = 0.$$

We note that the homogeneous equation for the above equation is the modified Bessel differential equation. Hence the solution is given as a superposition of $I_n(kr), K_n(kr)$. Here, I_n, K_n are the modified Bessel functions of the first and second kinds, respectively. We obtain

$$g_n(r_x,r_y) = K_n\left(k\max(r_x,r_y)\right)I_n\left(k\min(r_x,r_y)\right) - \frac{K_n(kR)+k\ell K_n'(kR)}{I_n(kR)+k\ell I_n'(kR)} I_n(kr_x)I_n(kr_y).$$

We note

$$I_n'(x) = \frac{1}{2}\left(I_{n-1}(x)+I_{n+1}(x)\right), \quad K_n'(x) = -\frac{1}{2}\left(K_{n-1}(x)+K_{n+1}(x)\right), \quad x\in\mathbb{R}.$$

Hence,

$$u_0(x) = \int_0^{2\pi}\int_0^R G(x,y)f(r_y,\theta_y)r_y\,\mathrm{d}r_y\mathrm{d}\theta_y = e^{i\alpha\theta_x}g_\alpha(r_x,R).$$

We have

$$g_\alpha(R,R) = I_\alpha(kR)K_\alpha(kR) - d_\alpha I_\alpha(kR),$$

where

$$d_\alpha = \frac{K_\alpha(kR)+k\ell K_\alpha'(kR)}{I_\alpha(kR)+k\ell I_\alpha'(kR)} I_\alpha(kR).$$

For the forward data, we observe u, u_0 at $r_x = R$, $\theta_x = 0$. That is, the outgoing light is measured at one point on the boundary while boundary values were observed at different points on the boundary in [26]. See appendix for the calculation of u. Let us set $M_D = 1$ (i.e. $M_{\mathrm{SD}} = M_S$). For the vector $\boldsymbol{\psi}\in\mathbb{R}^{M_{\mathrm{SD}}}$, we have

$$\psi_\alpha = \ln\frac{\left(K_\alpha(kR)-d_\alpha\right)I_\alpha(kR)}{I_\alpha(kR)K_\alpha(kR)+b_\alpha K_\alpha(kR)+c_\alpha I_\alpha(kR)} \tag{3}$$

for $\alpha = 1,\ldots,M_{\mathrm{SD}}$. We note that b_α, c_α, which are given in appendix, depend on η_a, R_a, k, R, ℓ.

Let us introduce

$$G^{(n)}(r_x,r_y) := g_n(r_x,r_y)r_y.$$

We obtain

$$\begin{aligned}&\left(K_j\eta^{\otimes j}\right)(x)\\ &= (-1)^{j+1}g^j e^{i\alpha\theta_x}\int_0^R\cdots\int_0^R G^{(\alpha)}(R,r_{y_1})G^{(\alpha)}(r_{y_1},r_{y_2})\cdots G^{(\alpha)}(r_{y_{j-1}},r_{y_j})G^{(\alpha)}(r_{y_j},R)\\ &\quad\times\eta(r_{y_1})\cdots\eta(r_{y_j})\,\mathrm{d}r_{y_1}\cdots\mathrm{d}r_{y_j},\quad x\in\Gamma,\end{aligned}$$

and

$$\begin{aligned}\left(\check{K}_j\eta^{\otimes j}\right)(x) &= \left(\frac{1}{u_0}K_j\eta^{\otimes j}\right)(x)\\ &= \frac{(-1)^{j+1}g^j}{G^{(\alpha)}(R,R)}\int_0^R\cdots\int_0^R G^{(\alpha)}(R,r_{y_1})G^{(\alpha)}(r_{y_1},r_{y_2})\cdots G^{(\alpha)}(r_{y_{j-1}},r_{y_j})G^{(\alpha)}(r_{y_j},R)\\ &\quad\times\eta(r_{y_1})\cdots\eta(r_{y_j})\,\mathrm{d}r_{y_1}\cdots\mathrm{d}r_{y_j},\quad x\in\Gamma.\end{aligned}$$

5.3. Implementation of the inverse Rytov series

Let us begin by writing

$$\psi_\alpha = \sum_{j=1}^{\infty} \left(J_j^{(\alpha)} \eta^{\otimes j} \right)(R), \quad \alpha = 1, \ldots, M_{\mathrm{SD}}.$$

We consider how the jth-order operator $\mathcal{J}_j$ in the inverse Rytov series can be numerically constructed. Here we assume that $r \in (0, R)$ is discretized into N_r points r_i ($i = 1, \ldots, N_r$) with small interval Δr. Thus, η can be expressed by a vector $\boldsymbol{\eta} \in \mathbb{R}^{N_r}$.

5.3.1. Forward vectors.

We set

$$r_i = i\Delta r \quad (i = 1, \ldots, N_r), \quad \Delta r = \frac{R}{N_r}.$$

Let $\mathbf{b} \in \mathbb{R}^{N_r}$ be a vector. We define $\mathbf{K}_0 \in \mathbb{R}^{M_{\mathrm{SD}}}$, $\mathbf{K}_1 \in \mathbb{R}^{M_{\mathrm{SD}} N_r}$ as

$$\{\mathbf{K}_0\}_\alpha = -G^{(\alpha)}(R, R),$$

$$\{\mathbf{K}_1(\mathbf{b})\}_{i+(\alpha-1)N_r} = g\Delta r \sum_{n=1}^{N_r} G^{(\alpha)}(r_i, r_n) G^{(\alpha)}(r_n, R) \{\mathbf{b}\}_n$$

for $1 \leqslant \alpha \leqslant M_{\mathrm{SD}}$, $1 \leqslant i \leqslant N_r$. Moreover,

$$\{\mathbf{K}_j(\mathbf{b}_1, \ldots, \mathbf{b}_j)\}_{i+(\alpha-1)N_r} = -g\Delta r \sum_{n=1}^{N_r} G^{(\alpha)}(r_i, r_n) \{\mathbf{b}_j\}_n \{\mathbf{K}_{j-1}(\mathbf{b}_1, \ldots, \mathbf{b}_{j-1})\}_{n+(\alpha-1)N_r}.$$

Using there $\mathbf{K}_j \in \mathbb{R}^{M_{\mathrm{SD}} N_r}$, we introduce

$$\{\mathbf{J}_j(\mathbf{b}_1, \ldots, \mathbf{b}_j)\}_\alpha = \sum_{m=1}^{j} \frac{(-1)^m}{m \{\mathbf{K}_0\}_\alpha^m} \times \sum_{i_1 + \cdots i_m = j} \{\mathbf{K}_{i_1}(\mathbf{b}_1, \ldots, \mathbf{b}_{i_1})\}_{\alpha N_r} \cdots \{\mathbf{K}_{i_m}(\mathbf{b}_{j-i_m+1}, \ldots, \mathbf{b}_j)\}_{\alpha N_r}$$

for $\alpha = 1, \ldots, M_{\mathrm{SD}}$.

5.3.2. Linearized problem.

In particular, we have

$$\{\mathbf{J}_1(\mathbf{b})\}_\alpha = -\frac{1}{\{\mathbf{K}_0\}_\alpha} \{\mathbf{K}_1(\mathbf{b})\}_{\alpha N_r}$$

for $\alpha = 1, \ldots, M_{\mathrm{SD}}$, $i = 1, \ldots, N_r$. From this, we can define a matrix $\underline{J}_1 \in \mathbb{R}^{M_{\mathrm{SD}} \times N_r}$ such that $\mathbf{J}_1(\mathbf{b}) = \underline{J}_1 \mathbf{b}$ as

$$\{\underline{J}_1\}_{\alpha,i} = \frac{g\Delta r}{G^{(\alpha)}(R, R)} \left[G^{(\alpha)}(R, r_i) \right]^2.$$

Using $\underline{J}_1$, we compute $\underline{\mathcal{J}}_1$. Here, $\underline{\mathcal{J}}_1$ is the Moore–Penrose pseudoinverse with a regularizer such as the truncated singular value decomposition:

$$\underline{\mathcal{J}}_1 = \underline{J}_{1,\mathrm{reg}}^{+} \in \mathbb{R}^{N_r \times M_{\mathrm{SD}}}.$$

The first term of the inverse Rytov series can be calculated as

$$\boldsymbol{\eta}_1 = \underline{\mathcal{J}}_1 \boldsymbol{\psi},$$

where

$$\{\boldsymbol{\eta}_1\}_i = \eta_1(r_i), \quad i = 1, \ldots, N_r.$$

We solve $\boldsymbol{\psi} = \underline{J}_1 \boldsymbol{\eta}_1$ as follows.

5.3.2.1. Underdetermined. Suppose we have

$$M_{\mathrm{SD}} \leqslant N_r.$$

That is, the inverse problem is underdetermined.

In this case, we obtain

$$\boldsymbol{\eta}_1 = \underline{J}_{1,\mathrm{reg}}^{+} \boldsymbol{\psi},$$

where

$$\underline{J}_{1,\mathrm{reg}}^{+} = \underline{J}_1^{*} \underline{M}_{\mathrm{reg}}^{-1}, \quad \underline{M} = \underline{J}_1 \underline{J}_1^{*}.$$

Here, $*$ denotes the Hermitian conjugate and reg means that the pesudoinverse is regularized by discarding singular values that are smaller than σ_0. Let σ_j^2 and $\mathbf{v}_j$ be the eigenvalues and eigenvectors of the matrix $\underline{M}$:

$$\underline{M}\mathbf{z}_j = \sigma_j^2 \mathbf{z}_j.$$

We obtain

$$\boldsymbol{\eta}_1 = \sum_{\substack{j \\ \sigma_j > \sigma_0}} \frac{1}{\sigma_j^2} \left(\mathbf{z}_j^{*} \boldsymbol{\psi}\right) \underline{J}_1^{*} \mathbf{z}_j.$$

5.3.2.2. Overdetermined. Suppose we have

$$M_{\mathrm{SD}} \geqslant N_r.$$

That is, the inverse problem is overdetermined.

In this case, we obtain

$$\boldsymbol{\eta}_1 = \underline{J}_{1,\mathrm{reg}}^{+} \boldsymbol{\psi},$$

where

$$\underline{J}_{1,\mathrm{reg}}^{+} = \underline{M}_{\mathrm{reg}}^{-1} \underline{J}_1^{*}, \quad \underline{M} = \underline{J}_1^{*} \underline{J}_1.$$

After solving the eigenproblem $\underline{M}\mathbf{z}_j = \sigma_j^2 \mathbf{z}_j$, we obtain

$$\boldsymbol{\eta}_1 = \sum_{\substack{j \\ \sigma_j > \sigma_0}} \frac{1}{\sigma_j^2} \left(\mathbf{z}_j^{*} \underline{J}_1^{*} \boldsymbol{\psi}\right) \mathbf{z}_j.$$

5.3.3. Inversion. Let $\mathbf{a}_1, \ldots, \mathbf{a}_j$ be real vectors of dimension M_{SD}. To compute the jth-order term $\boldsymbol{\eta}_j$, let us first introduce

$$\boldsymbol{\eta}_i^{(1)} = \underline{\mathcal{J}}_1 \mathbf{a}_i \quad (i = 1, \ldots, j).$$

We introduce vector $\mathcal{J}_j(\mathbf{a}_1,\ldots,\mathbf{a}_j)\in\mathbb{R}^{N_r}$ which has a recursive structure: for $j=1$,

$$\mathcal{J}_1(\mathbf{a}_1)=\boldsymbol{\eta}_1^{(1)},$$

and for $j\geqslant 2$,

$$\mathcal{J}_j(\mathbf{a}_1,\ldots,\mathbf{a}_j)\\ =-\sum_{m=1}^{j-1}\sum_{i_1+\cdots+i_m=j}\mathcal{J}_m\left(\mathbf{J}_{i_1}(\boldsymbol{\eta}_1^{(1)},\ldots,\boldsymbol{\eta}_{i_1}^{(1)}),\ldots,\mathbf{J}_{i_m}(\boldsymbol{\eta}_{j-i_m+1}^{(1)},\ldots,\boldsymbol{\eta}_j^{(1)})\right).$$

More specifically, $\mathcal{J}_j(\mathbf{a}_1,\ldots,\mathbf{a}_j)$ can be computed as follows. If $j=1$, then $\boldsymbol{\eta}_1^{(1)}$ is returned. For $j\geqslant 2$, we let m move from 1 to $j-1$. We form the compositions $[i_1,\ldots,i_m]$ such that $i_1+\cdots+i_m=j$. For each m $(1\leqslant m\leqslant j-1)$ and each composition $(i_1,\ldots,i_m)$, we compute

$$\boldsymbol{\eta}_{\rm tmp}=-\mathcal{J}_m\left(\mathbf{J}_{i_1}(\boldsymbol{\eta}_1^{(1)},\ldots,\boldsymbol{\eta}_{i_1}^{(1)}),\ldots,\mathbf{J}_{i_m}(\boldsymbol{\eta}_{j-i_m+1}^{(1)},\ldots,\boldsymbol{\eta}_j^{(1)})\right).$$

Let $\boldsymbol{\Sigma}(m)$ denote the sum of $\boldsymbol{\eta}_{\rm tmp}$ for all $\begin{pmatrix} j-1\\ m-1\end{pmatrix}$ compositions. The above step is repeated for all m $(1\leqslant m\leqslant j-1)$. We obtain

$$\mathcal{J}_j(\mathbf{a}_1,\ldots,\mathbf{a}_j)=\sum_{m=1}^{j-1}\boldsymbol{\Sigma}(m).$$

The jth term is calculated as

$$\boldsymbol{\eta}_j=\mathcal{J}_j(\boldsymbol{\psi},\ldots,\boldsymbol{\psi}).$$

In this way, we obtain $\boldsymbol{\eta}_j$ $(j=1,\ldots,N)$. The Nth-order approximation is given by

$$\boldsymbol{\eta}^{(N)}=\boldsymbol{\eta}_1+\cdots+\boldsymbol{\eta}_N.$$

Finally, the reconstruction can be done as follows. We have $\mu_a(r)=g(1+\eta(r))$. The reconstructed $\mu_a(r)$ is obtained as

$$\mu_a(r_i)\approx\mu_a^{(N)}(r_i)=g\left(1+\left\{\boldsymbol{\eta}^{(N)}\right\}_i\right).$$

5.4. Numerical results

We set $k=1$, $R=3$, $R_a=1.5$, $\ell=0.3$. Moreover, $N_r=M_{\rm SD}=90$. We chose σ_0 such that the largest 23 singular values were taken. Since only 23 singular values are taken, η is not fully reconstructed. When reconstructing η, we obtain at most $\boldsymbol{\eta}_{\rm proj}\in\mathbb{R}^{N_r}$, which is given by

$$\boldsymbol{\eta}_{\rm proj}=\underline{\mathcal{J}}_1\underline{J}_1\boldsymbol{\eta}.$$

In figures 1 through 3, $\boldsymbol{\eta},\boldsymbol{\eta}_{\rm proj},\boldsymbol{\eta}^{(1)},\boldsymbol{\eta}^{(2)},\boldsymbol{\eta}^{(3)},\boldsymbol{\eta}^{(4)},\boldsymbol{\eta}^{(5)}$ are shown.

In figure 1, $\eta_a=0.2$ (Left) and 1.0 (Right), and in figure 2, $\eta_a=2.0$ (Left) and 5.0 (Right). In the case of $\eta_a=0.2$, the first Rytov approximation $\boldsymbol{\eta}^{(1)}$ is different from $\boldsymbol{\eta}_{\rm proj}$ but the third Rytov approximation $\boldsymbol{\eta}^{(3)}$ already gives a good reconstruction. For $\eta_a=1.0$, the reconstruction approaches $\eta_{\rm proj}$ after the fifth term $\boldsymbol{\eta}_5$ is added. When $\eta_a=2.0$, the reconstruction is reasonable after $\boldsymbol{\eta}_5$ is added but still different from $\boldsymbol{\eta}_{\rm proj}$. When η_a is large and $\eta_a=5.0$, all reconstructions differ from $\boldsymbol{\eta}_{\rm proj}$.

Noise was added for figure 3. For both u_0,u, Gaussian noise with mean zero was added. The standard deviation of the noise was the standard deviation of u_0 multiplied by a constant γ. For u_0,u, no noise was added when the resulting value became negative. Figure 3 shows the

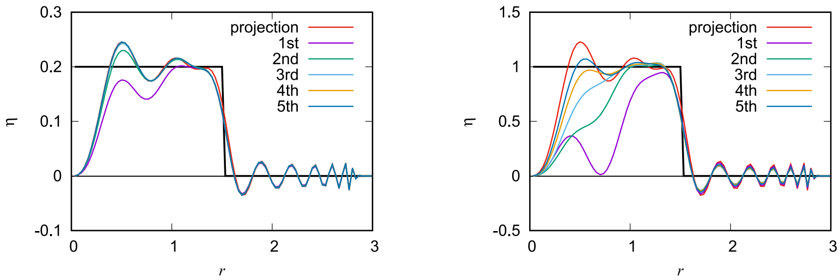

Figure 1. Reconstruction of η. The forward data is given in (3). We set (Left) $\eta_a = 0.2$ and (Right) $\eta_a = 1$.

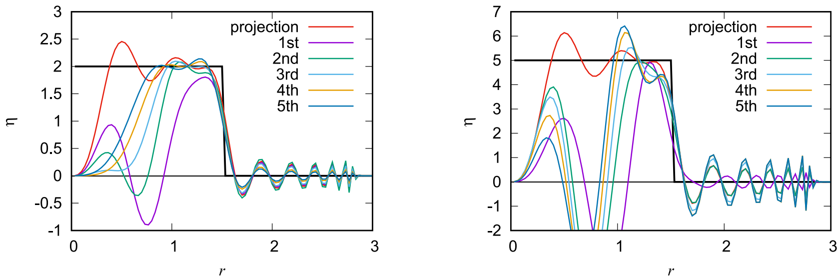

Figure 2. Reconstruction of η. The forward data is given in (3). We set (Left) $\eta_a = 2$, and (Right) $\eta_a = 5$.

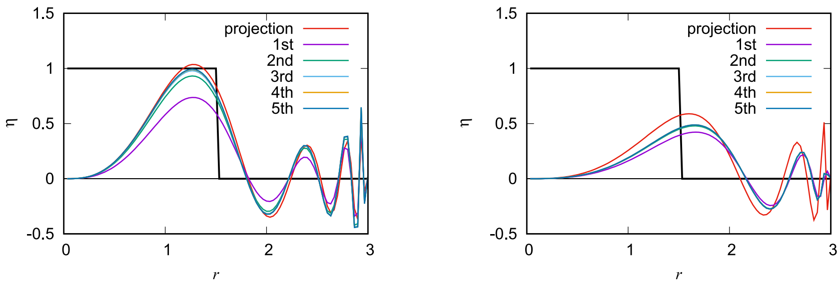

Figure 3. Reconstruction of η when $\eta_a = 1.0$. Gaussian noise with (Left) $\gamma = 10^{-4}$ and (Right) $\gamma = 10^{-5}$ was added. The largest 9 and 7 singular values were used for the weaker and stronger noise levels, respectively.

reconstruction of η for $\eta_a = 1.0$. Due to noise, fewer numbers of singular values had to be used. The largest nine singular values were used for $\gamma = 10^{-4}$ and the largest seven singular values were used when $\gamma = 10^{-5}$.

6. Concluding remarks

In this paper, multilinear forward operators $J_j : L^q(B_a) \times \cdots \times L^q(B_a) \to L^p(\Gamma)$ and inverse operators $\mathcal{J}_j : L^p(\Gamma) \times \cdots \times L^p(\Gamma) \to L^q(B_a)$ were considered. As was done for the inverse

Born series [4, 11, 16, 27], it is possible to consider the inverse Rytov series for nonlinear inverse problems in Banach spaces X,Y, for which the forward problem is from X to Y instead of from $L^q(B_a)$ to $L^p(\Gamma)$.

Although the expression of ψ_j in the Rytov series is more complicated than that of u_j in the Born series, the inverse Rytov series can also be computed in a recursive manner.

In this paper, the diffusion coefficient D_0 was assumed to be a known constant. Markel and Schotland has discussed the simultaneous reconstruction of the two functions with the (first) Rytov approximation [20]. It is an interesting future issue to extend the inverse Rytov series to the case of simultaneous reconstruction.

Data availability statement

The numerical values for the figures will not be on a website but can be available upon request. The data that support the findings of this study are available upon reasonable request from the authors. https://dataverse.harvard.edu/dataset.xhtml?persistentId=doi:10.7910/DVN/0X4QFP.

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Appendix. Forward data

Let G_a be the Green's function of the two-dimensional radial problem for the equation in which $\eta=\eta_a$ in $r\in[0,R_a]$ and $\eta=0$ otherwise. In the case of the delta-function source $\delta(x-x_s)$, $x_s\in\partial\Omega$, we have [26]

$$G_a(x,x_s)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}a_n e^{in(\theta-\theta_s)}I_n(k_a r),\qquad r\in\Omega_1,$$

and

$$\begin{aligned}G_a(x,x_s)=&\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}e^{in(\theta-\theta_s)}I_n(kr)K_n(kR)\\&+\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}e^{in(\theta-\theta_s)}\left(b_nK_n(kr)+c_nI_n(kr)\right),\qquad r\in\Omega_2.\end{aligned}$$

Here, coefficients a_n,b_n,c_n can be computed as the solution to the following system of linear equations, which is derived from the interface and boundary conditions.

$$\begin{pmatrix} I_n(k_aR_a) & -K_n(kR_a) & -I_n(kR_a)\\ k_aI_n'(k_aR_a) & -kK_n'(kR_a) & -kI_n'(kR_a)\\ 0 & K_n(kR)+k\ell K_n'(kR) & I_n(kR)+k\ell I_n'(kR)\end{pmatrix}\begin{pmatrix}a_n\\ b_n\\ c_n\end{pmatrix}$$
$$=\begin{pmatrix} I_n(kR_a)K_n(kR)\\ kI_n'(kR_a)K_n(kR)\\ k\ell I_n(kR)K_n'(kR)+I_n(kR)K_n(kR)\end{pmatrix}.$$

Therefore we obtain for $x \in \partial\Omega$,

$$u(x) = \int_0^{2\pi} \int_0^R G_a(x,y) f(r_y, \theta_y) r_y \, \mathrm{d}r_y \mathrm{d}\theta_y$$
$$= Re^{i\alpha\theta_x} \left[I_\alpha(kR) K_\alpha(kR) + b_\alpha K_\alpha(kR) + c_\alpha I_\alpha(kR) \right].$$

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References

[1] Abhishek A, Bonnet M and Moskow S 2020 Modified forward and inverse Born series for the Calderon and diffuse-wave problems *Inverse Problems* **36** 114001
[2] Arridge S R 1999 Optical tomography in medical imaging *Inverse Problems* **15** R41–R93
[3] Arridge S, Moskow S and Schotland J C 2012 Inverse Born series for the Calderon problem *Inverse Problems* **28** 035003
[4] Bardsley P and Vasquez F G 2014 Restarted inverse Born series for the Schrödinger problem with discrete internal measurements *Inverse Problems* **30** 045014
[5] Boas D A 1997 A fundamental limitation of linearized algorithms for diffuse optical tomography *Opt. Express* **1** 404–13
[6] Choe R *et al* 2005 Diffuse optical tomography of breast cancer during neoadjuvant chemotherapy: a case study with comparison to MRI *Med. Phys.* **32** 1128–39
[7] Choe R *et al* 2009 Differentiation of benign and malignant breast tumors by *in-vivo* three-dimensional parallel-plate diffuse optical tomography *J. Biomed. Opt.* **14** 024020
[8] Chung F J, Gilbert A C, Hoskins J G and Schotland J C 2017 Optical tomography on graphs *Inverse Problems* **33** 055016
[9] Eggebrecht A T, Ferradal S L, Robichaux-Viehoever A, Hassanpour M S, Dehghani H, Snyder A Z, Hershey T and Culver J P 2014 Mapping distributed brain function and networks with diffuse optical tomography *Nat. Photon.* **8** 448–54
[10] Engl H W, Hanke M and Neubauer A 1996 *Regularization of Inverse Problems* (Kluwer Academic Publishers Group)
[11] Hoskins J G and Schotland J C 2022 Analysis of the inverse Born series: an approach through geometric function theory *Inverse Problems* **38** 074001
[12] Keller J B 1969 Accuracy and validity of the Born and Rytov approximations *J. Opt. Soc. Am.* **59** 1003–4
[13] Kilgore K, Moskow S and Schotland J C 2012 Inverse Born series for scalar waves *J. Comput. Math.* **30** 601–14
[14] Kilgore K, Moskow S and Schotland J C 2017 Convergence of the Born and inverse Born series for electromagnetic scattering *Appl. Anal.* **96** 1737–48
[15] Kirkinis E 2008 Renormalization group interpretation of the Born and Rytov approximations *J. Opt. Soc. Am.* A **25** 2499–508
[16] Lakhal A 2018 A direct method for nonlinear ill-posed problems *Inverse Problems* **34** 025002
[17] Machida M and Schotland J C 2015 Inverse Born series for the radiative transport equation *Inverse Problems* **31** 095009
[18] Markel V, O'Sullivan J and Schotland J C 2003 Inverse problem in optical diffusion tomography. IV. Nonlinear inversion formulas *J. Opt. Soc. Am.* A **20** 903–12
[19] Markel V A and Schotland J C 2002 Inverse problem in optical diffusion tomography. II. Role of boundary conditions *J. Opt. Soc. Am.* A **19** 558–66
[20] Markel V A and Schotland J C 2004 Symmetries, inversion formulas and image reconstruction for optical tomography *Phys. Rev.* E **70** 056616
[21] Markel V and Schotland J C 2007 On the convergence of the Born series in optical tomography with diffuse light *Inverse Problems* **23** 1445–65
[22] Markel V and Schotland J C 2022 Reduced inverse Born series: a computational study *J. Opt. Soc. Am.* A **39** C179–89

[23] Marks D L A 2006 Family of approximations spanning the Born and Rytov scattering series *Opt. Express* **14** 8837–48
[24] Morozov V A 1993 *Regularization Methods for Ill-Posed Problems* (CRC Press)
[25] Moskow S and Schotland J C 2008 Convergence and stability of the inverse scattering series for diffuse waves *Inverse Problems* **24** 065005
[26] Moskow S and Schotland J C 2009 Numerical studies of the inverse Born series for diffuse waves *Inverse Problems* **25** 095007
[27] Moskow S and Schotland J C 2019 Inverse Born series *The Radon Transform* (*Radon Series on Computational and Applied Mathematics* vol 22) ed R Ramlau and O Scherzer (De Gruyter) ch 12
[28] Panasyuk G Y, Markel V A, Carney P S and Schotland J C 2006 Nonlinear inverse scattering and three-dimensional near-field optical imaging *Appl. Phys. Lett.* **89** 221116
[29] Park S, de Hoop M V, Calandra H and Shin C 2011 Full waveform inversion: a diffuse optical tomography point of view *SEG Technical Program Expanded Abstracts* vol 30 pp 2471–5
[30] Schuster T, Kaltenbacher B, Hofmann B and Kazimierski K 2012 *Regularization Methods in Banach Spaces* (De Gruyter)
[31] Shehadeh H A H, Malcolm A E and Schotland J C 2017 Inversion of the Bremmer series *J. Comput. Math.* **35** 586–99
[32] Tsihrintzis G A and Devaney A J 2000 Higher order (nonlinear) diffraction tomography: inversion of the Rytov series *IEEE Trans. Inf. Theory* **46** 1748–61