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LIPSCHITZ STABILITY IN INVERSE SOURCE AND INVERSE COEFFICIENT PROBLEMS FOR A FIRST- AND HALF-ORDER TIME-FRACTIONAL DIFFUSION EQUATION*

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Abstract. We consider inverse problems for the first- and half-order time-fractional equation. We establish the stability estimates of Lipschitz type in inverse source and inverse coefficient problems by means of the Carleman estimates.

Key words. fractional diffusion equation, inverse problem, Carleman estimate, stability estimate

AMS subject classifications. 35R11, 35R30

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1. Introduction.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the boundary $\partial\Omega$ of C^2 class. We set $Q = \Omega \times (0,T)$, where $T > 0$. We use notation $\partial_t = \frac{\partial}{\partial t}$, $\partial_i = \frac{\partial}{\partial x_i}$ $(i = 1,2,\dots,n)$. We also use the multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ with $\alpha_j \in \mathbb{N} \cup \{0\}$ $(j = 1,2,\dots,n)$, $\partial_x^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. Let $\nu = \nu(x)$ be the outward unit normal vector to $\partial\Omega$ at x and let $\partial_\nu = \nu \cdot \nabla$. In general, the βth-order Caputo-type fractional derivative is defined by

$$\partial_t^\beta u(x,t) := \frac{1}{\Gamma(n-\beta)} \int_0^t \frac{1}{(t-\tau)^{\beta+1-n}} \frac{\partial^n u(x,\tau)}{\partial \tau^n}\, d\tau, \quad (x,t) \in Q,$$

for $n-1 < \beta < n$, $n \in \mathbb{N}$ (see, e.g., [8, 31]). Here, Γ is the gamma function. We consider the following first- and half-order time-fractional diffusion equation:

$$\left(\rho_1\partial_t + \rho_2\partial_t^{\frac{1}{2}} - A\right) u(x,t) = g(x,t), \qquad (x,t) \in Q, \tag{1.1}$$

$$u(x,t) = h_1(x,t), \qquad (x,t) \in \partial\Omega \times (0,T), \tag{1.2}$$

$$u(x,0) = h_2(x), \qquad x \in \Omega, \tag{1.3}$$

where $\rho_1 > 0$, $\rho_2 \neq 0$ are constants, and A is a symmetric uniformly elliptic operator given by

$$Au(x,t) := \sum_{i,j=1}^n \partial_i(a_{ij}(x)\partial_j u(x,t)) - \sum_{j=1}^n b_j(x)\partial_j u(x,t) - c(x)u(x,t),\ (x,t) \in Q.$$

We assume that $a_{ij} \in C^3(\overline{\Omega})$, $a_{ij} = a_{ji}$ $(i,j = 1,2,\dots,n)$, $b_j \in C^2(\overline{\Omega})$ $(j =$

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$1, 2, \ldots, n)$, $c \in C^2(\overline{\Omega})$, and moreover there exists a constant $\mu > 0$ such that

$$\frac{1}{\mu}|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \mu|\xi|^2, \quad \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n,\ x \in \overline{\Omega}.$$

In fluid dynamics, (1.1) appears in the Basset problem [4] when the motion of a particle in a nonuniform flow is considered [6, 21, 30]. The first- and half-order time-fractional equation (1.1) also appears in porous media. Starting with the microscopic diffusion in a heterogeneous medium which has two length scales, the microscopic length scale of a typical porous block and the relative fracture width, a diffusion equation with the first- and half-order time derivatives is obtained at the large scale limit by the homogenization process [1, 2].

The first- and half-order equation (1.1) is one of parabolic equations with multiple time-fractional terms, i.e., the time-derivative part in the equation is given by $\sum_{j=0}^{\ell} p_j\partial_t^{\alpha_j}$, where $0 < \alpha_\ell < \cdots < \alpha_1 < \alpha_0 \le 1$ and coefficients p_j generally depend on x. Initial-boundary-value problems for multiterm time-fractional diffusion equations were considered in [29]. In the case that all time derivatives are noninteger order and the time-derivative part is given by $\sum_{j=1}^{\ell} p_j\partial_t^{\alpha_j}$, the well-posedness was investigated [23] and moreover the uniqueness in inverse boundary-value problems was proven [24]. An exact solution was obtained in the special case of a two-term time-fractional diffusion equation [5]. The uniqueness for two kinds of inverse problems of identifying fractional orders in diffusion equations with multiple time-fractional derivatives was proved [25]. The uniqueness in determining the spatial component of the source term from interior observation was established [14]. The maximum principle and uniqueness was considered in [27] for the determination of the temporal component of the source term from a single point observation. Also the unique continuation was considered for multiterm time-fractional diffusion equations [26].

In [16], the Hölder stability is proven for the inverse source problem of (1.1) (see also [22]). In this paper, we further prove the Lipshitz stability not only for the inverse source problem but also for the inverse coefficient problem for (1.1).

The methodology of our stability analysis is based on the technique of the Carleman estimate [9], which was pioneered by Bukhgeim and Klibanov [7] when they proved the global uniqueness in inverse problems. See also [17, 18], recent reviews [19, 35], and textbooks [13, 20]. The Carleman estimate is a weighted L^2 inequality for a solution of a partial differential equation. In the case of parabolic equations with one first-order time derivative, the global Lipschitz stability was proven by using this method of Carleman estimates [12]. In this paper we make use of the Carleman estimate for parabolic equations. The Carleman estimates have been used for differential equations with a single term time-fractional derivative [15, 34, 36].

This paper is organized as follows. In section 2, inverse source problems are considered. In section 3, inverse coefficient problems are considered. In section 4, the Carleman estimate necessary for our paper is established. Finally, proofs of the main theorems are given in section 5.

2. Inverse source problems. We consider the inverse problems of determining the time-independent source factor of (1.1) from spatial data and two types of observations. One is the boundary observation and the other is the interior observation.

Let $t_0 \in (0, T)$ be an arbitrarily fixed time. Let γ be an arbitrarily fixed open connected subboundary of $\partial\Omega$ and let ω be an arbitrary fixed subdomain of Ω such that $\omega \Subset \Omega$. We set $\Sigma = \gamma \times (0, T)$ and $Q_\omega = \omega \times (0, T)$.

Moreover we choose $\delta > 0$ such that

$$0 < t_0 - \delta < t_0 < t_0 + \delta < T,$$

and we set $Q_\delta = \Omega \times (t_0 - \delta, t_0 + \delta)$, $\Sigma_\delta = \gamma \times (t_0 - \delta, t_0 + \delta)$, $Q_{\omega,\delta} = \omega \times (t_0 - \delta, t_0 + \delta)$.

We assume that

$$\begin{cases} R \in C([0,T]; C(\overline{\Omega})) \cap C^2((0,T); C^2(\overline{\Omega})) \cap C^3((0,T); C(\overline{\Omega})), \\ \partial_t^{\frac{1}{2}} R \in C^2((0,T); C(\overline{\Omega})) \text{ and } |R(x,t_0)| > 0, \ x \in \overline{\Omega}. \end{cases} \tag{2.1}$$

Furthermore we define

$$\mathcal{U} = L^2(0,T; H^4(\Omega)) \cap H^1(0,T; H^2(\Omega)) \cap H^2(0,T; L^2(\Omega)).$$

Let us assume that $g(x,t)$ in (1.1) has the form

$$g(x,t) = f(x)R(x,t) \tag{2.2}$$

and set $h_1 = 0$ in Q, $h_2 = 0$ in Ω.

We consider

$$\left(\rho_1 \partial_t + \rho_2 \partial_t^{\frac{1}{2}} - A\right) u(x,t) = f(x)R(x,t), \qquad (x,t) \in Q, \tag{2.3}$$

$$u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T), \tag{2.4}$$

$$u(x,0) = 0, \qquad x \in \Omega, \tag{2.5}$$

and investigate two kinds of inverse problems depending on the type of observation.

In the inverse source problem via boundary observation, we determine $f(x)$, $x \in \Omega$, by spatial data $u(x,t_0)$, $x \in \Omega$, and boundary data on Σ. In the inverse source problem via interior observation, we determine $f(x)$, $x \in \Omega$, by spatial data $u(x,t_0)$, $x \in \Omega$, and interior data in Q_ω. The main theorems, Theorems 2.1 and 2.2 are stated as follows.

THEOREM 2.1. *Let us assume that* $u, \partial_t u, \partial_t^2 u \in \mathcal{U}$ *and* u *satisfies* (2.3)–(2.5). *We suppose that* $f \in H^2(\Omega)$ *with* $f = 0$ *on* $\partial\Omega$ *and* $\nabla f = 0$ *on* γ *and* R *satisfies* (2.1). *Then there exist constants* $C > 0$ *such that*

$$\|f\|_{H^2(\Omega)} \le C\|u(\cdot,t_0)\|_{H^4(\Omega)} + CB, \tag{2.6}$$

where

$$B = \|\nabla \partial_t^3 u\|_{L^2(\Sigma_\delta)} + \|\nabla \partial_t^2 u\|_{L^2(\Sigma_\delta)} + \|\nabla \partial_t u\|_{L^2(\Sigma_\delta)}.$$

THEOREM 2.2. *Let us assume that* $u, \partial_t u, \partial_t^2 u \in \mathcal{U}$ *and* u *satisfies* (2.3)–(2.5). *We suppose that* $f \in H^2(\Omega)$ *with* $f = 0$ *on* $\partial\Omega$ *and* $f = 0$ *in* ω *and* R *satisfies* (2.1). *Then there exist constants* $C > 0$ *such that*

$$\|f\|_{H^2(\Omega)} \le C\|u(\cdot,t_0)\|_{H^4(\Omega)} + CI, \tag{2.7}$$

where

$$I = \|\partial_t^3 u\|_{L^2(Q_{\omega,\delta})} + \|\partial_t^2 u\|_{L^2(Q_{\omega,\delta})} + \|\partial_t u\|_{L^2(Q_{\omega,\delta})}.$$

Remark 2.3. There is another approach to obtain the Lipschitz stability in inverse source problems by final observation data. In [33], Sakamoto and Yamamoto considered the perturbation of the single term time-fractional diffusion equations with a parameter as the diffusion coefficient and they obtained the stability estimate by means of the analytic perturbation theory under the appropriate assumptions on the parameter. In our case, however, we may not adopt their methodology directly since we consider the diffusion coefficient without the perturbation.

3. Inverse coefficient problems.

3.1. Determination of the zeroth-order coefficient. Let us consider the inverse problem of determining the zeroth-order coefficient. In (1.1), we consider two coefficients $c^{(k)}(x)$, $x \in \Omega$ $(k = 1, 2)$, where $c^{(k)} \in C^2(\overline{\Omega})$, $c^{(k)}(x) \geq 0$, $x \in \Omega$ $(k = 1, 2)$. We assume that there exists a constant $M_1 > 0$ such that

$$\|c^{(k)}\|_{C^2(\overline{\Omega})} \leq M_1 \tag{3.1}$$

for $k = 1, 2$. Let $u^{(k)}(x, t)$ be the corresponding solutions. We write A as

$$Au^{(k)}(x, t) = A_0^{(k)} u^{(k)}(x, t), \tag{3.2}$$

where $A_0^{(k)}$ is defined as

$$\begin{aligned} A_0^{(k)} u(x, t) = &\sum_{i,j=1}^{n} \partial_i(a_{ij}(x)\partial_j u^{(k)}(x, t)) \\ &- \sum_{j=1}^{n} b_j(x)\partial_j u^{(k)}(x, t) - c^{(k)}(x)u^{(k)}(x, t),\ (x, t) \in Q, \end{aligned}$$

for $k = 1, 2$.

By subtraction we obtain

$$\begin{cases} \left(\rho_1\partial_t + \rho_2\partial_t^{\frac{1}{2}} - A_0^{(1)}\right) u(x, t) = f(x)R(x, t), & (x, t) \in Q, \\ u(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ u(x, 0) = 0, & x \in \Omega, \end{cases}$$

where

$$f(x) = c^{(1)}(x) - c^{(2)}(x), \quad u(x, t) = u^{(1)}(x, t) - u^{(2)}(x, t), \quad R(x, t) = -u^{(2)}(x, t)$$

for $(x, t) \in Q$. Thus we arrive at the following two theorems as a direct consequence of the inverse source problems. In both cases the Lipschitz stability is obtained. Theorem 3.1 is proved using the inverse source problem via boundary observation stated in Theorem 2.1. Theorem 3.2 is proved using the inverse source problem via interior observation stated in Theorem 2.2.

THEOREM 3.1 (boundary observation). *Let $u^{(k)}, \partial_t u^{(k)}, \partial_t^2 u^{(k)} \in \mathcal{U}$ $(k = 1, 2)$, and $u^{(1)}, u^{(2)}$ satisfy* (1.1)–(1.3) *with* (3.2). *We suppose that $c^{(1)}, c^{(2)} \in C^2(\overline{\Omega})$ satisfy* (3.1) *with $c^{(1)} = c^{(2)}$ on $\partial\Omega$ and $\nabla c^{(1)} = \nabla c^{(2)}$ on γ, and $R = -u^{(2)}$ satisfies* (2.1). *Then there exist constants $C > 0$ such that*

$$\|c^{(1)} - c^{(2)}\|_{H^2(\Omega)} \leq C\|u^{(1)}(\cdot, t_0) - u^{(2)}(\cdot, t_0)\|_{H^4(\Omega)} + CB, \tag{3.3}$$

where

$$B = \|\nabla\partial_t^3(u^{(1)} - u^{(2)})\|_{L^2(\Sigma_\delta)} + \|\nabla\partial_t^2(u^{(1)} - u^{(2)})\|_{L^2(\Sigma_\delta)} + \|\nabla\partial_t(u^{(1)} - u^{(2)})\|_{L^2(\Sigma_\delta)}.$$

THEOREM 3.2 (interior observation). *Let $u^{(k)}, \partial_t u^{(k)}, \partial_t^2 u^{(k)} \in \mathcal{U}$ $(k = 1, 2)$, and $u^{(1)}, u^{(2)}$ satisfy* (1.1)–(1.3) *with* (3.2). *We suppose that $c^{(1)}, c^{(2)} \in C^2(\overline{\Omega})$ satisfy* (3.1) *with $c^{(1)} = c^{(2)}$ in $\partial\Omega \cup \omega$ and $R = -u^{(2)}$ satisfies* (2.1). *Then there exist constants $C > 0$ such that*

$$\|c^{(1)} - c^{(2)}\|_{H^2(\Omega)} \le C\|u^{(1)}(\cdot, t_0) - u^{(2)}(\cdot, t_0)\|_{H^4(\Omega)} + CI, \tag{3.4}$$

where

$$I = \|\partial_t^3(u^{(1)} - u^{(2)})\|_{L^2(Q_{\omega,\delta})} + \|\partial_t^2(u^{(1)} - u^{(2)})\|_{L^2(Q_{\omega,\delta})} + \|\partial_t(u^{(1)} - u^{(2)})\|_{L^2(Q_{\omega,\delta})}.$$

Remark 3.3. In the case of diffusion in porous media, the condition $|u^{(2)}(x, t_0)| = |R(x, t_0)| > 0$ for $x \in \overline{\Omega}$ means that the concentration of tracer particles is nonzero.

Remark 3.4. We may determine the first-order coefficients b_j $(j = 1, \dots, n)$ by n observations via the similar argument used in the proofs of Theorems 2.1 and 2.2.

3.2. Determination of the diffusion coefficient. We consider diffusion coefficients $a^{(k)}$ $(k = 1, 2)$ and corresponding solutions $u^{(k)}$. Let us express A as

$$Au^{(k)}(x, t) = A_2^{(k)} u^{(k)}(x, t), \tag{3.5}$$

where $A_2^{(k)}$ is defined as

$$A_2^{(k)} u^{(k)}(x, t) = \operatorname{div}(a^{(k)}(x)\nabla u^{(k)}(x, t)) - \mathbf{b}(x)\cdot\nabla u^{(k)}(x, t) - c(x)u^{(k)}(x, t), \quad (x, t) \in Q,$$

for $k = 1, 2$. We suppose that $a^{(k)} \in C^4(\overline{\Omega})$ $(k = 1, 2)$, $\mathbf{b} = (b_1, b_2, \dots, b_n) \in \{C^3(\overline{\Omega})\}^n$, and $c \in C^3(\overline{\Omega})$. Moreover we assume that there exist constants $m > 0$ and $M_2 > 0$ such that

$$a^{(k)}(x) \ge m, \ x \in \Omega, \quad \text{and} \quad \|a^{(k)}\|_{C^3(\overline{\Omega})} \le M_2 \tag{3.6}$$

for $k = 1, 2$. We investigate the inverse problems of determining the diffusion coefficients $a^{(k)}$ $(k = 1, 2)$ by boundary observations and interior observations.

Set

$$a(x) = a^{(1)}(x) - a^{(2)}(x), \quad u(x, t) = u^{(1)}(x, t) - u^{(2)}(x, t), \quad r(x, t) = u^{(2)}(x, t)$$

for $(x, t) \in Q$. Then by subtracting the equations for $k = 2$ from ones for $k = 1$, we obtain

$$\begin{cases} \left(\rho_1\partial_t + \rho_2\partial_t^{\frac{1}{2}} - A_2^{(1)}\right) u(x, t) = \operatorname{div}(a(x)\nabla r(x, t)), & (x, t) \in Q, \\ u(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ u(x, 0) = 0, & x \in \Omega. \end{cases} \tag{3.7}$$

We assume that

$$\begin{cases} r \in C([0, T]; C^3(\overline{\Omega})) \cap C((0, T); C^5(\overline{\Omega})) \\ \qquad \cap \ C^2((0, T); C^4(\overline{\Omega})) \cap C^3((0, T); C^2(\overline{\Omega})), \\ \partial_t^{\frac{1}{2}} r \in C((0, T); C^3(\overline{\Omega})) \cap C^2((0, T); C^2(\overline{\Omega})). \end{cases} \tag{3.8}$$

Let us introduce weight functions for the Carleman estimates introduced in section 4. Since we consider two types of observation, we prepare two kinds of distance functions d_1 and d_2. For the symmetric uniformly elliptic operator A, we choose $d_1 \in C^2(\overline{\Omega})$ such that

$$d_1(x) > 0,\ x \in \Omega, \quad |\nabla d_1(x)| > \sigma_1,\ x \in \overline{\Omega},$$
$$\sum_{i,j=1}^{n} a_{ij}(x)\partial_i d_1 \nu_j \le 0,\ x \in \partial\Omega \setminus \gamma,$$

where $\sigma_1 > 0$ is a constant. Let ω_0 be an arbitrarily fixed subdomain of Ω such that $\omega_0 \Subset \omega$. We take $d_2 \in C^2(\overline{\Omega})$ such that

$$d_2(x) > 0,\ x \in \Omega, \quad |\nabla d_2(x)| > \sigma_2,\ x \in \overline{\Omega \setminus \omega_0}, \quad d_2(x) = 0,\ x \in \partial\Omega,$$

where $\sigma_2 > 0$ is a constant. The existence of the distance functions d_1 and d_2 is proved in [10, 11, 12]. Then we introduce weight functions φ_k, ψ_k $(k = 1, 2)$ as

$$\varphi_k(x,t) = \frac{e^{\lambda d_k(x)}}{t(T-t)}, \quad \psi_k(x,t) = \frac{e^{\lambda d_k(x)} - e^{2\lambda \|d_k\|_{C(\overline{\Omega})}}}{t(T-t)}, \quad (x,t) \in Q.$$

For the distance functions d_1 and d_2, we assume that there exists a constant $m_1 > 0$ such that

$$|\nabla r(x,t_0) \cdot \nabla d_1(x)| \ge m_1, \quad x \in \overline{\Omega}, \tag{3.9}$$

or that there exists a constant $m_2 > 0$ such that

$$|\nabla r(x,t_0) \cdot \nabla d_2(x)| \ge m_2, \quad x \in \overline{\Omega \setminus \omega}. \tag{3.10}$$

Let D' be an arbitrary subdomain such that $\omega \Subset D' \Subset \Omega$. Set $D = \Omega \setminus D'$. Henceforth we suppose that $a \equiv 0$ in D.

Now we are ready to state our main results.

THEOREM 3.5 (boundary observation). *Let $u^{(k)}, \partial_t u^{(k)}, \partial_t^2 u^{(k)} \in \mathcal{U}$, $\nabla u^{(k)} \in \mathcal{U}^n$ $(k = 1, 2)$, and $u^{(1)}, u^{(2)}$ satisfy* (1.1)–(1.3) *with* (3.5). *We suppose that $a^{(1)}, a^{(2)} \in C^4(\overline{\Omega})$ satisfy* (3.6) *with $a^{(1)} = a^{(2)}$ in D and $r = u^{(2)}$ satisfies* (3.8) *and* (3.9). *Then there exist constants $C > 0$ such that*

$$\|a^{(1)} - a^{(2)}\|_{H^3(\Omega)} \le C\|u^{(1)}(\cdot,t_0) - u^{(2)}(\cdot,t_0)\|_{H^5(\Omega)} + CB, \tag{3.11}$$

where

$$B = \|\nabla\partial_t^3(u^{(1)} - u^{(2)})\|_{L^2(\Sigma_\delta)} + \|\nabla\partial_t^2(u^{(1)} - u^{(2)})\|_{L^2(\Sigma_\delta)} + \|\nabla\partial_t(u^{(1)} - u^{(2)})\|_{L^2(\Sigma_\delta)}.$$

THEOREM 3.6 (interior observation). *Let $u^{(k)}, \partial_t u^{(k)}, \partial_t^2 u^{(k)} \in \mathcal{U}$, $\nabla u^{(k)} \in \mathcal{U}^n$ $(k = 1, 2)$, and $u^{(1)}, u^{(2)}$ satisfy* (1.1)–(1.3) *with* (3.5). *We suppose that $a^{(1)}, a^{(2)} \in C^4(\overline{\Omega})$ satisfy* (3.6) *with $a^{(1)} = a^{(2)}$ in $D \cup \omega$ and $r = u^{(2)}$ satisfies* (3.8) *and* (3.10). *Then there exist constants $C > 0$ such that*

$$\|a^{(1)} - a^{(2)}\|_{H^3(\Omega)} \le C\|u^{(1)}(\cdot,t_0) - u^{(2)}(\cdot,t_0)\|_{H^5(\Omega)} + CI, \tag{3.12}$$

where

$$I = \|\partial_t^3(u^{(1)} - u^{(2)})\|_{L^2(Q_{\omega,\delta})} + \|\partial_t^2(u^{(1)} - u^{(2)})\|_{L^2(Q_{\omega,\delta})} + \|\partial_t(u^{(1)} - u^{(2)})\|_{L^2(Q_{\omega,\delta})}.$$

Remark 3.7. In the case of one spatial dimension, we may relax some assumptions on $u^{(k)}$ $(k = 1, 2)$. It depends on the assumptions of the Carleman estimate for the third-order partial differential equations (Lemmas 4.8 and 4.9). See also [32].

3.3. Simultaneous determination of the diffusion coefficient and zeroth-order coefficient. Let us consider diffusion coefficients $a^{(k)}$ $(k = 1, 2)$, zeroth-order coefficient $c^{(k)}$, and corresponding solutions $u_\ell^{(k)}$ $(\ell = 1, 2)$. We express $\mathcal{A}^{(k)}$ as

$$Au_\ell^{(k)}(x,t) = \mathcal{A}^{(k)} u_\ell^{(k)}(x,t), \tag{3.13}$$

where $\mathcal{A}^{(k)}$ is defined as

$$\mathcal{A}^{(k)} u_\ell^{(k)}(x,t) = \operatorname{div}(a^{(k)}(x)\nabla u_\ell^{(k)}(x,t)) - c^{(k)}(x) u_\ell^{(k)}(x,t), \quad (x,t) \in Q,$$

for $k = 1, 2$. We suppose that $a^{(k)} \in C^4(\overline{\Omega})$ and $c^{(k)} \in C^3(\overline{\Omega})$ $(k = 1, 2)$. We investigate the inverse problems of determining $a^{(k)}$ and $c^{(k)}$ $(k = 1, 2)$ by boundary and interior observations for two inputs, $\ell = 1, 2$.

We consider

$$\left(\rho_1 \partial_t + \rho_2 \partial_t^{\frac{1}{2}} - \mathcal{A}^{(k)}\right) u_\ell^{(k)}(x,t) = g_\ell(x,t), \qquad (x,t) \in Q, \tag{3.14}$$

$$u_\ell^{(k)}(x,t) = h_{1,\ell}(x,t), \qquad (x,t) \in \partial\Omega \times (0,T), \tag{3.15}$$

$$u_\ell^{(k)}(x,0) = h_{2,\ell}(x), \qquad x \in \Omega, \tag{3.16}$$

for $k = 1, 2$, $\ell = 1, 2$.

Set

$$\begin{aligned} &a(x) = a^{(1)}(x) - a^{(2)}(x), \quad c(x) = c^{(1)}(x) - c^{(2)}(x), \\ &u_\ell(x,t) = u_\ell^{(1)}(x,t) - u_\ell^{(2)}(x,t), \quad r_\ell(x,t) = u_\ell^{(2)}(x,t) \end{aligned}$$

for $(x,t) \in Q$, $\ell = 1, 2$. Then by subtracting the equations for $k = 2$ from ones for $k = 1$, we obtain

$$\begin{cases} \left(\rho_1 \partial_t + \rho_2 \partial_t^{\frac{1}{2}} - \mathcal{A}^{(1)}\right) u_\ell(x,t) & \\ \qquad = \operatorname{div}(a(x)\nabla r_\ell(x,t)) - c(x) r_\ell(x,t), & (x,t) \in Q, \\ u_\ell(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u_\ell(x,0) = 0, & x \in \Omega, \end{cases} \tag{3.17}$$

for $\ell = 1, 2$.

We assume that there exists a constant $m_3 > 0$ such that

$$\left|\Big(r_2(x,t_0)\nabla r_1(x,t_0) - r_1(x,t_0)\nabla r_2(x,t_0)\Big) \cdot \nabla d_1(x)\right| \geq m_3, \quad x \in \overline{\Omega}, \tag{3.18}$$

or that there exists a constant $m_4 > 0$ such that

$$\left|\Big(r_2(x,t_0)\nabla r_1(x,t_0) - r_1(x,t_0)\nabla r_2(x,t_0)\Big) \cdot \nabla d_2(x)\right| \geq m_4, \quad x \in \overline{\Omega \setminus \omega}. \tag{3.19}$$

Let D' be an arbitrary subdomain such that $\omega \Subset D' \Subset \Omega$. Set $D = \Omega \setminus D'$.

Theorem 3.8 (boundary observation). *Let $u^{(k)}, \partial_t u^{(k)}, \partial_t^2 u^{(k)} \in \mathcal{U}$, $\nabla u^{(k)} \in \mathcal{U}^n$ $(k=1,2)$, and $u^{(1)}, u^{(2)}$ satisfy* (3.14)–(3.16) *with* (3.13). *We suppose that $a^{(1)}, a^{(2)} \in C^4(\overline{\Omega})$ satisfy* (3.6) *with $a^{(1)} = a^{(2)}$ in D, $c^{(1)}, c^{(2)} \in C^2(\overline{\Omega})$ satisfy* (3.1) *with $c^{(1)} = c^{(2)}$ on $\partial\Omega$ and $\nabla c^{(1)} = \nabla c^{(2)}$ on γ, and $r = u^{(2)}$ satisfies* (3.8), (3.18) *and $|r_1(x,t_0)| > 0$, $x \in \overline{\Omega}$. Then there exist constants $C > 0$ such that*

(3.20)
$$\|a^{(1)} - a^{(2)}\|_{H^3(\Omega)} + \|c^{(1)} - c^{(2)}\|_{H^2(\Omega)} \le C \sum_{\ell=1}^{2} \|u_\ell^{(1)}(\cdot,t_0) - u_\ell^{(2)}(\cdot,t_0)\|_{H^5(\Omega)} + CB,$$

where

$$B = \sum_{\ell=1}^{2} \Bigg(\|\nabla \partial_t^3 (u_\ell^{(1)} - u_\ell^{(2)})\|_{L^2(\Sigma_\delta)} + \|\nabla \partial_t^2 (u_\ell^{(1)} - u_\ell^{(2)})\|_{L^2(\Sigma_\delta)} + \|\nabla \partial_t (u_\ell^{(1)} - u_\ell^{(2)})\|_{L^2(\Sigma_\delta)} \Bigg).$$

Theorem 3.9 (interior observation). *Let $u^{(k)}, \partial_t u^{(k)}, \partial_t^2 u^{(k)} \in \mathcal{U}$, $\nabla u^{(k)} \in \mathcal{U}^n$ $(k=1,2)$, and $u^{(1)}, u^{(2)}$ satisfy* (3.14)–(3.16) *with* (3.13). *We suppose that $a^{(1)}, a^{(2)} \in C^4(\overline{\Omega})$ satisfy* (3.6) *with $a^{(1)} = a^{(2)}$ in $D \cup \omega$, $c^{(1)}, c^{(2)} \in C^2(\overline{\Omega})$ satisfy* (3.1) *with $c^{(1)} = c^{(2)}$ in $\partial\Omega \cup \omega$, and $r = u^{(2)}$ satisfies* (3.8), (3.19) *and $|r_1(x,t_0)| > 0$, $x \in \overline{\Omega}$. Then there exist constants $C > 0$ such that*

$$\|a^{(1)} - a^{(2)}\|_{H^3(\Omega)} + \|c^{(1)} - c^{(2)}\|_{H^2(\Omega)} \le \sum_{\ell=1}^{2} C \|u_\ell^{(1)}(\cdot,t_0) - u_\ell^{(2)}(\cdot,t_0)\|_{H^5(\Omega)} + CI, \tag{3.21}$$

where

$$I = \sum_{\ell=1}^{2} \Bigg(\|\partial_t^3 (u_\ell^{(1)} - u_\ell^{(2)})\|_{L^2(Q_{\omega,\delta})} + \|\partial_t^2 (u_\ell^{(1)} - u_\ell^{(2)})\|_{L^2(Q_{\omega,\delta})} + \|\partial_t (u_\ell^{(1)} - u_\ell^{(2)})\|_{L^2(Q_{\omega,\delta})} \Bigg).$$

4. Carleman estimate. In this section, we establish the Carleman estimates for (1.1). We transform (1.1) into an integer-order partial differential equation. The calculation is similar to [34]. Let us begin with the following lemma.

Lemma 4.1 (Lemma 3.1 in [16]). *If $u \in C([0,T]; H^4(\Omega)) \cap C^1((0,T); H^2(\Omega)) \cap C^2((0,T); L^2(\Omega))$ satisfies* (1.1) *through* (1.3), *then u satisfies*

$$\rho_2^2 \partial_t u(x,t) - (\rho_1 \partial_t - A)^2 u(x,t) = G(x,t), \quad (x,t) \in Q, \tag{4.1}$$

where

$$G(x,t) = \left[\rho_2 \partial_t^{\frac{1}{2}} - (\rho_1 \partial_t - A) \right] g(x,t) + \frac{\rho_2 g(x,0)}{\sqrt{\pi t}}, \quad (x,t) \in Q. \tag{4.2}$$

Although $\partial_t^{\frac{1}{2}} \partial_t^{\frac{1}{2}} \neq \partial_t$ in general, we may obtain the above lemma by applying $\rho_2 \partial_t^{\frac{1}{2}} - (\rho_1 \partial_t - A)$ to both sides of (1.1) and using $u(x,0) = 0$, $x \in \Omega$.

Now we are ready to state our Carleman estimates.

Theorem 4.2 (Carleman estimate for (1.1) with boundary data). *Let $p \ge 0$. Suppose that $g(x,t) = 0$, $(x,t) \in \partial\Omega \times (0,T)$, and $\nabla g(x,t) = 0$, $(x,t) \in \Sigma$. Then there exists $\lambda_0 > 0$ such that for any $\lambda > \lambda_0$, we can choose $s_0(\lambda) > 0$ for which there* *exists* $C = C(s_0, \lambda) > 0$ *such that*

$$\begin{aligned}(4.3)\quad &\int_Q \Bigg[(s\varphi_1)^{p-1} \left(|\partial_t^2 u|^2 + \sum_{i,j=1}^n |\partial_t \partial_i \partial_j u|^2 \right) + (s\varphi_1)^{p+1} |\nabla \partial_t u|^2 \\ &\quad + (s\varphi_1)^{p+2} |\nabla(\rho_1 \partial_t - A) u|^2 + (s\varphi_1)^{p+3} \left(|\partial_t u|^2 + \sum_{i,j=1}^n |\partial_i \partial_j u|^2 \right) \\ &\quad + (s\varphi_1)^{p+5} |\nabla u|^2 + (s\varphi_1)^{p+7} |u|^2 \Bigg] e^{2s\psi_1}\, dxdt \\ &\le C \int_Q (s\varphi_1)^{p+1} \left| \left[\rho_2^2 \partial_t - (\rho_1 \partial_t - A)^2 \right] u \right|^2 e^{2s\psi_1}\, dxdt \\ &\quad + C \int_\Sigma \left[(s\varphi_1)^{p+1} |\nabla \partial_t u|^2 + (s\varphi_1)^{p+2} |\nabla \partial_t^{\frac{1}{2}} u|^2 + (s\varphi_1)^{p+5} |\nabla u|^2 \right] e^{2s\psi_1}\, dSdt\end{aligned}$$

for all $s > s_0$ *and all* $u \in \mathcal{U}$ *satisfying* (1.1) *with* $u(x,t) = 0$, $(x,t) \in \partial\Omega \times (0,T)$, *and* $u(x,0) = 0$, $x \in \Omega$.

THEOREM 4.3 (Carleman estimate for (1.1) with interior data). *Let* $p \ge 0$. *Suppose that* $g(x,t)= 0$, $(x,t) \in \partial\Omega \times (0,T)$, *and* $g(x,t) = 0$, $(x,t) \in Q_\omega$. *Then there exists* $\lambda_0 > 0$ *such that for any* $\lambda > \lambda_0$, *we can choose* $s_0(\lambda) > 0$ *for which there exists* $C = C(s_0, \lambda) > 0$ *such that*

$$\begin{aligned}(4.4)\quad &\int_Q \Bigg[(s\varphi_2)^{p-1} \left(|\partial_t^2 u|^2 + \sum_{i,j=1}^n |\partial_t \partial_i \partial_j u|^2 \right) + (s\varphi_2)^{p+1} |\nabla \partial_t u|^2 \\ &\quad + (s\varphi_2)^{p+2} |\nabla(\rho_1 \partial_t - A) u|^2 + (s\varphi_2)^{p+3} \left(|\partial_t u|^2 + \sum_{i,j=1}^n |\partial_i \partial_j u|^2 \right) \\ &\quad + (s\varphi_2)^{p+5} |\nabla u|^2 + (s\varphi_2)^{p+7} |u|^2 \Bigg] e^{2s\psi_2}\, dxdt \\ &\le C \int_Q (s\varphi_2)^{p+1} \left| \left[\rho_2^2 \partial_t - (\rho_1 \partial_t - A)^2 \right] u \right|^2 e^{2s\psi_2}\, dxdt \\ &\quad + C \int_{Q_\omega} \left[(s\varphi_2)^{p+3} |\partial_t u|^2 + (s\varphi_2)^{p+4} |\partial_t^{\frac{1}{2}} u|^2 + (s\varphi_2)^{p+7} |u|^2 \right] e^{2s\psi_2}\, dxdt\end{aligned}$$

for all $s > s_0$ *and all* $u \in \mathcal{U}$ *satisfying* (1.1) *with* $u(x,t) = 0$, $(x,t) \in \partial\Omega \times (0,T)$, *and* $u(x,0) = 0$, $x \in \Omega$.

To prove Theorems 4.2 and 4.3, we start with the global Carleman estimates for parabolic equations (see, e.g., [11, 35]) stated in Lemmas 4.4 and 4.5 below.

LEMMA 4.4. *Let* $p \ge 0$. *There exists* $\lambda_0 > 0$ *such that for any* $\lambda > \lambda_0$, *we can choose* $s_0(\lambda) > 0$ *for which there exists* $C = C(s_0, \lambda) > 0$ *such that*

$$\begin{aligned}&\int_Q \left[(s\varphi_1)^{p-1} \left(|\partial_t v|^2 + \sum_{i,j=1}^n |\partial_i \partial_j v|^2 \right) + (s\varphi_1)^{p+1} |\nabla v|^2 + (s\varphi_1)^{p+3} |v|^2 \right] e^{2s\psi_1}\, dxdt \\ &\le C \int_Q (s\varphi_1)^p |(\rho_1 \partial_t - A) v|^2 e^{2s\psi_1}\, dxdt + C \int_\Sigma (s\varphi_1)^{p+1} |\nabla v|^2 e^{2s\psi_1}\, dSdt\end{aligned}$$

for all $s > s_0$ *and all* $v \in L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$ *satisfying* $v(x,t) = 0$, $(x,t) \in \partial\Omega \times (0,T)$.

LEMMA 4.5. *Let* $p \geq 0$. *There exists* $\lambda_0 > 0$ *such that for any* $\lambda > \lambda_0$, *we can choose* $s_0(\lambda) > 0$ *for which there exists* $C = C(s_0, \lambda) > 0$ *such that*

$$\int_Q \left[(s\varphi_2)^{p-1} \left(|\partial_t v|^2 + \sum_{i,j=1}^n |\partial_i \partial_j v|^2 \right) + (s\varphi_2)^{p+1} |\nabla v|^2 + (s\varphi_2)^{p+3} |v|^2 \right] e^{2s\psi_2} \, dxdt$$
$$\leq C \int_Q (s\varphi_2)^p |(\rho_1 \partial_t - A)v|^2 e^{2s\psi_2} \, dxdt + C \int_{Q_\omega} (s\varphi_2)^{p+3} |v|^2 e^{2s\psi_2} \, dxdt$$

for all $s > s_0$ *and all* $v \in L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$ *satisfying* $v(x,t) = 0$, $(x,t) \in \partial\Omega \times (0,T)$.

Proof of Theorem 4.2. Throughout the proof, we assume that $s > 1$ is large enough to satisfy $s\varphi > 1$ in Q.

Equation (4.1) yields

$$\rho_1 \partial_t w(x,t) - Aw(x,t) = \rho_2^2 \partial_t u(x,t) - G(x,t), \quad (x,t) \in Q, \tag{4.5}$$

where

$$w(x,t) = \rho_1 \partial_t u(x,t) - Au(x,t), \quad (x,t) \in Q. \tag{4.6}$$

Since $u(x,t) = 0$, $(x,t) \in \partial\Omega \times (0,T)$, and $g(x,t) = 0$, $(x,t) \in \partial\Omega \times (0,T)$, we have by (1.1),

$$w(x,t) = \rho_1 \partial_t u(x,t) - Au(x,t) = g(x,t) - \rho_2 \partial_t^{\frac{1}{2}} u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T).$$

Applying Lemma 4.4 to (4.5), we obtain

$$\begin{aligned} &\int_Q \left[(s\varphi_1)^{p_1-1} |\partial_t w|^2 + (s\varphi_1)^{p_1+1} |\nabla w|^2 + (s\varphi_1)^{p_1+3} |w|^2 \right] e^{2s\psi_1} \, dxdt \\ &\leq C \int_Q (s\varphi_1)^{p_1} |\partial_t u|^2 e^{2s\psi_1} \, dxdt + C \int_Q (s\varphi_1)^{p_1} |G|^2 e^{2s\psi_1} \, dxdt \\ &\quad + C \int_\Sigma (s\varphi_1)^{p_1+1} |\nabla w|^2 e^{2s\psi_1} \, dSdt \end{aligned} \tag{4.7}$$

for $p_1 \geq 0$. Next by applying Lemma 4.4 to (4.6), we obtain

$$\begin{aligned} &\int_Q \Bigg[(s\varphi_1)^{p_2-1} \left(|\partial_t u|^2 + \sum_{i,j=1}^n |\partial_i \partial_j u|^2 \right) \\ &\qquad + (s\varphi_1)^{p_2+1} |\nabla u|^2 + (s\varphi_1)^{p_2+3} |u|^2 \Bigg] e^{2s\psi_1} \, dxdt \\ &\leq C \int_Q (s\varphi_1)^{p_2} |w|^2 e^{2s\psi_1} \, dxdt + C \int_\Sigma (s\varphi_1)^{p_2+1} |\nabla u|^2 e^{2s\psi_1} \, dSdt \end{aligned} \tag{4.8}$$

for $p_2 \geq 0$.

Putting $p_2 = p_1 + 1$ and substituting the estimate of $|\partial_t u|^2$ in (4.8) into the right-hand side of (4.7), we obtain

$$\int_Q \left[(s\varphi_1)^{p_1-1}|\partial_t w|^2 + (s\varphi_1)^{p_1+1}|\nabla w|^2 + (s\varphi_1)^{p_1+3}|w|^2\right] e^{2s\psi_1}\, dxdt$$
$$\le C\int_Q (s\varphi_1)^{p_1+1}|w|^2 e^{2s\varphi_1}\, dxdt + C\int_Q (s\varphi_1)^{p_1}|G|^2 e^{2s\psi_1}\, dxdt + CB_{1,p_1},$$

where

$$B_{1,p_1} = \int_\Sigma \left[(s\varphi_1)^{p_1+1}|\nabla w|^2 + (s\varphi_1)^{p_1+2}|\nabla u|^2\right] e^{2s\psi_1}\, dSdt.$$

Taking sufficiently large $s > 0$, we can absorb the first term on the right-hand side of the above inequality into the left-hand side and we have

$$\int_Q \left[(s\varphi_1)^{p_1-1}|\partial_t w|^2 + (s\varphi_1)^{p_1+1}|\nabla w|^2 + (s\varphi_1)^{p_1+3}|w|^2\right] e^{2s\psi_1}\, dxdt$$
$$\le C\int_Q (s\varphi_1)^{p_1}|G|^2 e^{2s\psi_1}\, dxdt + CB_{1,p_1}. \tag{4.9}$$

By (4.8) with $p_2 = p_1 + 3$ and (4.9), we obitain

$$\int_Q \Bigg[(s\varphi_1)^{p_1+1}|\nabla(\rho_1\partial_t - A)u|^2 + (s\varphi_1)^{p_1+2}\left(|\partial_t u|^2 + \sum_{i,j=1}^n |\partial_i\partial_j u|^2\right)$$
$$+ (s\varphi_1)^{p_1+4}|\nabla u|^2 + (s\varphi_1)^{p_1+6}|u|^2\Bigg] e^{2s\psi_1}\, dxdt$$
$$\le C\int_Q (s\varphi_1)^{p_1}|G|^2 e^{2s\psi_1}\, dxdt + CB_{2,p_1}, \tag{4.10}$$

where

$$B_{2,p_1} = \int_\Sigma \left[(s\varphi_1)^{p_1+1}|\nabla w|^2 + (s\varphi_1)^{p_1+4}|\nabla u|^2\right] e^{2s\psi_1}\, dSdt.$$

Let us choose $p_1 = p + 1$ in (4.9). Then from (4.6) and (4.9), we have

$$\int_Q (s\varphi_1)^p|\partial_t(\rho_1\partial_t u - Au)|^2 e^{2s\psi_1}\, dxdt \le C\int_Q (s\varphi_1)^{p+1}|G|^2 e^{2s\psi_1}\, dxdt + CB_{1,p+1}.$$

Setting $u_0 = \partial_t u$, we obtain

$$\int_Q (s\varphi_1)^p|\rho_1\partial_t u_0 - Au_0|^2 e^{2s\psi_1}\, dxdt \le C\int_Q (s\varphi_1)^{p+1}|G|^2 e^{2s\psi_1}\, dxdt + CB_{1,p+1}. \tag{4.11}$$

If we use Lemma 4.4 with $v = u_0$ and applying (4.11), we obtain

$$\begin{aligned}
&\int_Q \Bigg[(s\varphi_1)^{p-1} \left(|\partial_t u_0|^2 + \sum_{i,j=1}^n |\partial_i \partial_j u_0|^2 \right) \\
&\qquad + (s\varphi_1)^{p+1} |\nabla u_0|^2 + (s\varphi_1)^{p+3} |u_0|^2 \Bigg] e^{2s\psi_1}\, dxdt \\
&\le C \int_Q (s\varphi_1)^{p+1} |G|^2 e^{2s\psi_1}\, dxdt + C \int_\Sigma (s\varphi_1)^{p+1} |\nabla u_0|^2 e^{2s\psi_1}\, dSdt + CB_{1,p+1}.
\end{aligned}$$

Recalling $u_0 = \partial_t u$, we have

$$\begin{aligned}
&\int_Q \Bigg[(s\varphi_1)^{p-1} \left(|\partial_t^2 u|^2 + \sum_{i,j=1}^n |\partial_t \partial_i \partial_j u|^2 \right) \\
&\qquad + (s\varphi_1)^{p+1} |\nabla \partial_t u|^2 + (s\varphi_1)^{p+3} |\partial_t u|^2 \Bigg] e^{2s\psi_1}\, dxdt \\
&\le C \int_Q (s\varphi_1)^{p+1} |G|^2 e^{2s\psi_1}\, dxdt + CB_{3,p},
\end{aligned}$$

where

$$B_{3,p} = \int_\Sigma \left[(s\varphi_1)^{p+1} |\nabla \partial_t u|^2 + (s\varphi_1)^{p+2} |\nabla w|^2 + (s\varphi_1)^{p+3} |\nabla u|^2 \right] e^{2s\psi_1}\, dSdt.$$

Hence using (4.10), we obtain

$$\begin{aligned}
&\int_Q \Bigg[(s\varphi_1)^{p-1} \left(|\partial_t^2 u|^2 + \sum_{i,j=1}^n |\partial_t \partial_i \partial_j u|^2 \right) + (s\varphi_1)^{p+1} |\nabla \partial_t u|^2 \\
&\qquad + (s\varphi_1)^{p+2} |\nabla (\rho_1 \partial_t - A) u|^2 + (s\varphi_1)^{p+3} \left(|\partial_t u|^2 + \sum_{i,j=1}^n |\partial_i \partial_j u|^2 \right) \\
&\qquad + (s\varphi_1)^{p+5} |\nabla u|^2 + (s\varphi_1)^{p+7} |u|^2 \Bigg] e^{2s\psi_1}\, dxdt \\
&\le C \int_Q (s\varphi_1)^{p+1} |G|^2 e^{2s\psi_1}\, dxdt + CB_{4,p},
\end{aligned}$$

where

$$B_{4,p} = \int_\Sigma \left[(s\varphi_1)^{p+1} |\nabla \partial_t u|^2 + (s\varphi_1)^{p+2} |\nabla w|^2 + (s\varphi_1)^{p+5} |\nabla u|^2 \right] e^{2s\psi_1}\, dSdt.$$

Finally, we consider the boundary term $B_{4,p}$. Since $\nabla g = 0$ on Σ is assumed, $\nabla w =$

$\nabla g - \rho_2 \nabla \partial_t^{\frac{1}{2}} u = -\rho_2 \nabla \partial_t^{\frac{1}{2}} u$ on Σ. Hence we have

$$\int_Q \Bigg[(s\varphi_1)^{p-1} \left(|\partial_t^2 u|^2 + \sum_{i,j=1}^n |\partial_t \partial_i \partial_j u|^2 \right) + (s\varphi_1)^{p+1} |\nabla \partial_t u|^2$$
$$+ (s\varphi_1)^{p+2} |\nabla(\rho_1 \partial_t - A)u|^2 + (s\varphi_1)^{p+3} \left(|\partial_t u|^2 + \sum_{i,j=1}^n |\partial_i \partial_j u|^2 \right)$$
$$+ (s\varphi_1)^{p+5} |\nabla u|^2 + (s\varphi_1)^{p+7} |u|^2 \Bigg] e^{2s\psi_1}\, dxdt$$
$$\le C \int_Q (s\varphi_1)^{p+1} |G|^2 e^{2s\psi_1}\, dxdt$$
$$+ C \int_\Sigma \left[(s\varphi_1)^{p+1} |\nabla \partial_t u|^2 + (s\varphi_1)^{p+2} |\nabla \partial_t^{\frac{1}{2}} u|^2 + (s\varphi_1)^{p+5} |\nabla u|^2 \right] e^{2s\psi_1}\, dSdt.$$

Thus we obtain (4.3). ∎

Proof of Theorem 4.3. By using Lemma 4.5 instead of Lemma 4.4, we can prove Theorem 4.3 in the same way as Theorem 4.2. ∎

Furthermore we need Carleman estimates for elliptic equations in the proof of the stability estimates in inverse source problems which we will develop in section 5.

Let us assume that $\widetilde{a}_{ij} \in C^1(\overline{\Omega})$, $\widetilde{a}_{ij} = \widetilde{a}_{ji}$ $(i,j = 1,\dots,n)$, $\widetilde{b}_j \in C(\overline{\Omega})$ $(j = 1,\dots,n)$, $\widetilde{c} \in C(\overline{\Omega})$, and that there exists a constant $\widetilde{\mu} > 0$ such that

$$\frac{1}{\widetilde{\mu}} |\xi|^2 \le \sum_{i,j=1}^n \widetilde{a}_{ij}(x) \xi_i \xi_j \le \widetilde{\mu} |\xi|^2, \quad \xi = (\xi_1,\dots,\xi_n) \in \mathbb{R}^n,\ x \in \overline{\Omega}.$$

We consider the following symmetric uniformly elliptic operator:

$$\widetilde{A}\widetilde{v}(x) := \sum_{i,j=1}^n \partial_i(\widetilde{a}_{ij}(x) \partial_j \widetilde{v}(x)) - \sum_{j=1}^n \widetilde{b}_j(x) \partial_j \widetilde{v}(x) - \widetilde{c}(x) \widetilde{v}(x),\ x \in \Omega.$$

Set $\widetilde{\varphi}_k(x) := \varphi_k(x, t_0)$, $x \in \Omega$, and $\widetilde{\psi}_k(x) := \psi_k(x, t_0)$, $x \in \Omega$, for $k = 1, 2$. Then we have the following lemmas.

LEMMA 4.6. *Let $p \ge 0$. There exists $\lambda_0 > 0$ such that for any $\lambda > \lambda_0$, we can choose $s_0(\lambda) > 0$ for which there exists $C = C(s_0, \lambda) > 0$ such that*

$$\int_\Omega \left[(s\widetilde{\varphi}_1)^{p-1} \sum_{i,j=1}^n |\partial_i \partial_j \widetilde{v}|^2 + (s\widetilde{\varphi}_1)^{p+1} |\nabla \widetilde{v}|^2 + (s\widetilde{\varphi}_1)^{p+3} |\widetilde{v}|^2 \right] e^{2s\widetilde{\psi}_1}\, dx$$
$$\le C \int_\Omega (s\widetilde{\varphi}_1)^p |\widetilde{A}\widetilde{v}|^2 e^{2s\widetilde{\psi}_1}\, dx + C \int_\gamma (s\widetilde{\varphi}_1)^{p+1} |\nabla \widetilde{v}|^2 e^{2s\widetilde{\psi}_1}\, dS$$

for all $s > s_0$ and all $\widetilde{v} \in H^2(\Omega)$ satisfying $\widetilde{v}(x) = 0$, $x \in \partial\Omega$.

LEMMA 4.7. *Let $p \ge 0$. There exists $\lambda_0 > 0$ such that for any $\lambda > \lambda_0$, we can choose $s_0(\lambda) > 0$ for which there exists $C = C(s_0, \lambda) > 0$ such that*

$$\int_\Omega \left[(s\widetilde{\varphi}_2)^{p-1} \sum_{i,j=1}^n |\partial_i\partial_j \widetilde{v}|^2 + (s\widetilde{\varphi}_2)^{p+1}|\nabla\widetilde{v}|^2 + (s\widetilde{\varphi}_2)^{p+3}|\widetilde{v}|^2 \right] e^{2s\widetilde{\psi}_2}\,dx$$
$$\le C\int_\Omega (s\widetilde{\varphi}_2)^p|\widetilde{A}\widetilde{v}|^2 e^{2s\widetilde{\psi}_2}\,dx + C\int_\omega (s\widetilde{\varphi}_2)^{p+3}|\widetilde{v}|^2 e^{2s\widetilde{\psi}_2}\,dx$$

for all $s > s_0$ and all $\widetilde{v} \in H^2(\Omega)$ satisfying $\widetilde{v}(x) = 0$, $x \in \partial\Omega$.

These lemmas can be shown in the same manner as the parabolic case by means of integration by parts. Hence we omit the proofs of these lemmas here.

We conclude this section by introducing Carleman estimates for the third-order partial differential equations which we use in the proof of the stability estimates in inverse problems of determining the diffusion coefficients.

Let $\mathbf{p} = (p_1, \dots, p_2) \in \{C^1(\overline{\Omega})\}^n$.

LEMMA 4.8. *We assume that there exists $m_1 > 0$ such that $|\mathbf{p}(x)\cdot\nabla d_1(x)| \ge m_1$, $x \in \overline{\Omega}$. Then there exists $\lambda_0 > 0$ such that for any $\lambda > \lambda_0$, we can choose $s_0(\lambda) > 0$ for which there exists $C = C(s_0, \lambda) > 0$ such that*

$$\int_\Omega \Bigg[s\widetilde{\varphi}_1 \sum_{i,j,k=1}^n |\partial_i\partial_j\partial_k\widetilde{v}|^2 + (s\widetilde{\varphi}_1)^2|\nabla\triangle\widetilde{v}|^2$$
$$+ (s\widetilde{\varphi}_1)^3 \sum_{i,j=1}^n |\partial_i\partial_j\widetilde{v}|^2 + (s\widetilde{\varphi}_1)^5\left(|\nabla\widetilde{v}|^2 + |\widetilde{v}|^2\right)\Bigg] e^{2s\widetilde{\psi}_1}\,dx$$
$$\le C\int_\Omega \left(|\nabla(\mathbf{p}\cdot\nabla\triangle\widetilde{v})|^2 + |\mathbf{p}\cdot\nabla\triangle\widetilde{v}|^2\right) e^{2s\widetilde{\psi}_1}\,dx$$

for all $s > s_0$ and all $\widetilde{v} \in H^4(\Omega)$ satisfying $|\widetilde{v}(x)| = |\nabla\widetilde{v}(x)| = |\triangle\widetilde{v}(x)| = |\nabla\triangle\widetilde{v}(x)| = 0$, $x \in \partial\Omega$, and $|\nabla\partial_k\widetilde{v}(x)| = 0$, $x \in \gamma$ $(k = 1, 2, \dots, n)$.

LEMMA 4.9. *We assume that there exists $m_2 > 0$ such that $|\mathbf{p}(x)\cdot\nabla d_2(x)| \ge m_2$, $x \in \overline{\Omega\setminus\omega}$. Then there exists $\lambda_0 > 0$ such that for any $\lambda > \lambda_0$, we can choose $s_0(\lambda) > 0$ for which there exists $C = C(s_0, \lambda) > 0$ such that*

$$\int_\Omega \Bigg[s\widetilde{\varphi}_2 \sum_{i,j,k=1}^n |\partial_i\partial_j\partial_k\widetilde{v}|^2 + (s\widetilde{\varphi}_2)^2|\nabla\triangle\widetilde{v}|^2$$
$$+ (s\widetilde{\varphi}_2)^3 \sum_{i,j=1}^n |\partial_i\partial_j\widetilde{v}|^2 + (s\widetilde{\varphi}_2)^5\left(|\nabla\widetilde{v}|^2 + |\widetilde{v}|^2\right)\Bigg] e^{2s\widetilde{\psi}_2}\,dx$$
$$\le C\int_\Omega \left(|\nabla(\mathbf{p}\cdot\nabla\triangle\widetilde{v})|^2 + |\mathbf{p}\cdot\nabla\triangle\widetilde{v}|^2\right) e^{2s\widetilde{\psi}_2}\,dx$$

for all $s > s_0$ and all $\widetilde{v} \in H^4(\Omega)$ satisfying $|\widetilde{v}(x)| = |\nabla\widetilde{v}(x)| = |\triangle\widetilde{v}(x)| = |\nabla\triangle\widetilde{v}(x)| = 0$, $x \in \partial\Omega$, and $\widetilde{v}(x) = 0$, $x \in \omega$.

To establish the Carleman estimate for $\mathbf{p}\cdot\nabla\triangle\widetilde{v}$, we start by proving the first-order partial differential equations $\mathbf{p}\cdot\nabla\widetilde{v}$.

LEMMA 4.10. *We assume that there exists $m_1 > 0$ such that $|\mathbf{p}(x)\cdot\nabla d_1(x)| \ge m_1$, $x \in \overline{\Omega}$. Then there exists $\lambda_0 > 0$ such that for any $\lambda > \lambda_0$, we can choose $s_0(\lambda) > 0$ for which there exists $C = C(s_0, \lambda) > 0$ such that*

$$\int_\Omega (s\widetilde{\varphi}_1)^2 \left(|\nabla \widetilde{v}|^2 + |\widetilde{v}|^2\right) e^{2s\widetilde{\psi}_1}\, dx \le C \int_\Omega \left(|\nabla(\mathbf{p}\cdot\nabla\widetilde{v})|^2 + |\mathbf{p}\cdot\nabla\widetilde{v}|^2\right) e^{2s\widetilde{\psi}_1}\, dx$$

for all $s > s_0$ and all $\widetilde{v} \in H^2(\Omega)$ satisfying $|\widetilde{v}(x)| = |\nabla\widetilde{v}(x)| = 0$, $x \in \partial\Omega$.

LEMMA 4.11. *We assume that there exists $m_2 > 0$ such that $|\mathbf{p}(x)\cdot\nabla d_2(x)| \ge m_2$, $x \in \overline{\Omega\setminus\omega}$. Then there exists $\lambda_0 > 0$ such that for any $\lambda > \lambda_0$, we can choose $s_0(\lambda) > 0$ for which there exists $C = C(s_0, \lambda) > 0$ such that*

$$\int_\Omega (s\widetilde{\varphi}_2)^2 \left(|\nabla \widetilde{v}|^2 + |\widetilde{v}|^2\right) e^{2s\widetilde{\psi}_2}\, dx \le C \int_\Omega \left(|\nabla(\mathbf{p}\cdot\nabla\widetilde{v})|^2 + |\mathbf{p}\cdot\nabla\widetilde{v}|^2\right) e^{2s\widetilde{\psi}_2}\, dx$$

for all $s > s_0$ and all $\widetilde{v} \in H^2(\Omega)$ satisfying $|\widetilde{v}(x)| = |\nabla\widetilde{v}(x)| = 0$, $x \in \partial\Omega$, and $\widetilde{v}(x) = 0$, $x \in \omega$.

Proof of Lemma 4.10. Setting $\widetilde{w} = \widetilde{v}e^{s\widetilde{\psi}_1}$ in Ω, we have

$$e^{s\widetilde{\psi}_1}(\mathbf{p}\cdot\nabla\widetilde{v}) = \mathbf{p}\cdot\nabla\widetilde{w} - s\lambda\widetilde{\varphi}_1(\mathbf{p}\cdot\nabla d_1)\widetilde{w} \quad \text{in } \Omega.$$

Taking the weighted L^2 norm, by integrating by parts we obtain

$$\begin{aligned}
&\int_\Omega |\mathbf{p}\cdot\nabla\widetilde{v}|^2 e^{2s\widetilde{\psi}_1}\, dx \\
&= \int_\Omega |\mathbf{p}\cdot\nabla\widetilde{w}|^2\, dx + \int_\Omega s^2\lambda^2\widetilde{\varphi}_1^2(\mathbf{p}\cdot\nabla d_1)^2|\widetilde{w}|^2\, dx \\
&\quad - 2\int_\Omega s\lambda\widetilde{\varphi}_1(\mathbf{p}\cdot\nabla d_1)\sum_{j=1}^n p_j\widetilde{w}\partial_j\widetilde{w}\, dx \\
&\ge \int_\Omega s^2\lambda^2\widetilde{\varphi}_1^2(\mathbf{p}\cdot\nabla d_1)^2|\widetilde{w}|^2\, dx - \int_\Omega s\lambda\widetilde{\varphi}_1(\mathbf{p}\cdot\nabla d_1)\sum_{j=1}^n p_j\partial_j(\widetilde{w})^2\, dx \\
&= \int_\Omega s^2\lambda^2\widetilde{\varphi}_1^2(\mathbf{p}\cdot\nabla d_1)^2|\widetilde{w}|^2\, dx + \int_\Omega s\lambda^2\widetilde{\varphi}_1(\mathbf{p}\cdot\nabla d_1)^2|\widetilde{w}|^2\, dx \\
&\quad + \int_\Omega s\lambda\widetilde{\varphi}_1\left[(\mathbf{p}\cdot\nabla d_1)(\operatorname{div}\mathbf{p}) + \mathbf{p}\cdot\nabla(\mathbf{p}\cdot\nabla d_1)\right]|\widetilde{w}|^2\, dx.
\end{aligned}$$

Hence we have

$$\int_\Omega s^2\lambda^2\widetilde{\varphi}_1^2|\widetilde{w}|^2\, dx \le C\int_\Omega |\mathbf{p}\cdot\nabla\widetilde{v}|^2 e^{2s\widetilde{\psi}_1}\, dx + C\int_\Omega \left(s\lambda^2\widetilde{\varphi}_1 + s\lambda\widetilde{\varphi}_1\right)|\widetilde{w}|^2\, dx.$$

Taking sufficiently large $s > 0$, we may absorb the second term on the right-hand side of the above inequality into the left-hand side and we see that

$$\int_\Omega s^2\lambda^2\widetilde{\varphi}_1^2|\widetilde{w}|^2\, dx \le C\int_\Omega |\mathbf{p}\cdot\nabla\widetilde{v}|^2 e^{2s\widetilde{\psi}_1}\, dx,$$

that is,

$$\int_\Omega s^2\lambda^2\widetilde{\varphi}_1^2|\widetilde{v}|^2 e^{2s\widetilde{\psi}_1}\, dx \le C\int_\Omega |\mathbf{p}\cdot\nabla\widetilde{v}|^2 e^{2s\widetilde{\psi}_1}\, dx. \tag{4.12}$$

Set $\widetilde{v}_k = \partial_k \widetilde{v}$ in Ω for $k = 1, 2, \ldots, n$. We consider

$$\mathbf{p} \cdot \nabla \widetilde{v}_k = \partial_k (\mathbf{p} \cdot \nabla \widetilde{v}) - (\partial_k \mathbf{p}) \cdot \nabla \widetilde{v}.$$

Applying the estimate (4.12) to the above equation, we may obtain

$$\begin{aligned}\int_\Omega s^2 \lambda^2 \widetilde{\varphi}_1^2 |\widetilde{v}_k|^2 e^{2s\widetilde{\psi}_1}\, dx &\le C \int_\Omega |\mathbf{p} \cdot \nabla \widetilde{v}_k|^2 e^{2s\widetilde{\psi}_1}\, dx \\ &\le C \int_\Omega |\partial_k (\mathbf{p} \cdot \nabla \widetilde{v})|^2 e^{2s\widetilde{\psi}_1}\, dx + C \int_\Omega |(\partial_k \mathbf{p}) \cdot \nabla \widetilde{v}|^2 e^{2s\widetilde{\psi}_1}\, dx.\end{aligned}$$

Hence we have

$$\int_\Omega s^2 \lambda^2 \widetilde{\varphi}_1^2 |\nabla \widetilde{v}|^2 e^{2s\widetilde{\psi}_1}\, dx \le C \int_\Omega |\nabla (\mathbf{p} \cdot \nabla \widetilde{v})|^2 e^{2s\widetilde{\psi}_1}\, dx + C \int_\Omega |\nabla \widetilde{v}|^2 e^{2s\widetilde{\psi}_1}\, dx.$$

Choosing sufficiently large $s > 0$, we can absorb the second term on the right-hand side of the above inequality into the left-hand side and we may get

$$\int_\Omega s^2 \lambda^2 \widetilde{\varphi}_1^2 |\nabla \widetilde{v}|^2 e^{2s\widetilde{\psi}_1}\, dx \le C \int_\Omega |\nabla (\mathbf{p} \cdot \nabla \widetilde{v})|^2 e^{2s\widetilde{\psi}_1}\, dx.$$

Combining this with (4.12), we obtain the Carleman estimate of Lemma 4.10. Thus we conclude Lemma 4.10. ∎

Proof of Lemma 4.11. By an argument similar to that used in the proof of Lemma 4.10, we may obtain Lemma 4.11. ∎

Remark 4.12. In one spatial dimension, the assumption $|\nabla \widetilde{v}| = 0$ on $\partial\Omega$ is not necessary in Lemmas 4.10 and 4.11. In this case, we have the following Carleman estimate by integration by parts:

$$\int_\Omega (s\widetilde{\varphi}_k)^2 \left(|\partial_1 \widetilde{v}|^2 + |\widetilde{v}|^2 \right) e^{2s\widetilde{\psi}_k}\, dx \le C \int_\Omega |p_1 \partial_1 \widetilde{v}|^2 e^{2s\widetilde{\psi}_k}\, dx$$

for $k = 1, 2$.

Now we are ready to prove Lemmas 4.8 and 4.9.

Proof of Lemma 4.8. Set $\widetilde{y} = \triangle \widetilde{v}$ in Ω. By $|\triangle \widetilde{v}(x)| = |\nabla \triangle \widetilde{v}(x)| = 0$, $x \in \partial\Omega$, we see that $|\widetilde{y}(x)| = |\nabla \widetilde{y}(x)| = 0$, $x \in \partial\Omega$. By Lemma 4.10, we obtain

$$\int_\Omega (s\widetilde{\varphi}_1)^2 \left(|\nabla \widetilde{y}|^2 + |\widetilde{y}|^2 \right) e^{2s\widetilde{\psi}_1}\, dx \le C \int_\Omega \left(|\nabla (\mathbf{p} \cdot \nabla \widetilde{y})|^2 + |\mathbf{p} \cdot \nabla \widetilde{y}|^2 \right) e^{2s\widetilde{\psi}_1}\, dx,$$

that is,

$$\begin{aligned}\int_\Omega (s\widetilde{\varphi}_1)^2 &\left(|\nabla \triangle \widetilde{v}|^2 + |\triangle \widetilde{v}|^2 \right) e^{2s\widetilde{\psi}_1}\, dx \\ &\le C \int_\Omega \left(|\nabla (\mathbf{p} \cdot \nabla \triangle \widetilde{v})|^2 + |\mathbf{p} \cdot \nabla \triangle \widetilde{v}|^2 \right) e^{2s\widetilde{\psi}_1}\, dx.\end{aligned} \tag{4.13}$$

Next we use the Carleman estimate for elliptic equations to estimate the left-hand side of the above inequality. By Lemma 4.6 with $p = 2$, we have

$$\int_\Omega \left[s\widetilde{\varphi}_1 \sum_{i,j=1}^n |\partial_i\partial_j \widetilde{v}|^2 + (s\widetilde{\varphi}_1)^3 |\nabla \widetilde{v}|^2 + (s\widetilde{\varphi}_1)^5 |\widetilde{v}|^2 \right] e^{2s\widetilde{\psi}_1}\,dx \le C \int_\Omega (s\widetilde{\varphi}_1)^2 |\triangle \widetilde{v}|^2 e^{2s\widetilde{\psi}_1}\,dx. \tag{4.14}$$

Setting $\widetilde{v}_k = \partial_k \widetilde{v}$ in Ω for $k = 1, 2, \ldots, n$ and using Lemma 4.6 again, we see that

$$\int_\Omega \left[s\widetilde{\varphi}_1 \sum_{i,j=1}^n |\partial_i\partial_j \widetilde{v}_k|^2 + (s\widetilde{\varphi}_1)^3 |\nabla \widetilde{v}_k|^2 + (s\widetilde{\varphi}_1)^5 |\widetilde{v}_k|^2 \right] e^{2s\widetilde{\psi}_1}\,dx \le C \int_\Omega (s\widetilde{\varphi}_1)^2 |\triangle \widetilde{v}_k|^2 e^{2s\widetilde{\psi}_1}\,dx,$$

that is,

$$\int_\Omega \left[s\widetilde{\varphi}_1 \sum_{i,j,k=1}^n |\partial_i\partial_j\partial_k \widetilde{v}|^2 + (s\widetilde{\varphi}_1)^3 \sum_{i,j=1}^n |\partial_i\partial_j \widetilde{v}|^2 + (s\widetilde{\varphi}_1)^5 |\nabla\widetilde{v}|^2 \right] e^{2s\widetilde{\psi}_1}\,dx \le C \int_\Omega (s\widetilde{\varphi}_1)^2 |\nabla\triangle \widetilde{v}|^2 e^{2s\widetilde{\psi}_1}\,dx. \tag{4.15}$$

Summing up the inequalities (4.13) –(4.15), we may obtain the Carleman estimate of Lemma 4.8. ∎

Proof of Lemma 4.9. Using Lemmas 4.11 and 4.7 instead of Lemmas 4.10 and 4.6, we may prove Lemma 4.9 in the same way as Lemma 4.8. ∎

5. Proof of stability estimates. Hereafter we let C denote a generic constant which is independent of s, x, t and let $C(s)$ denote a generic constant which is independent of x, t but depends on s.

5.1. Stability in inverse source problems.

Proof of Theorem 2.1. By using Lemma 4.1 for (2.3), we obtain

$$\rho_2^2 \partial_t u(x,t) - (\rho_1\partial_t - A)^2 u(x,t) = F(x,t), \quad (x,t) \in Q, \tag{5.1}$$

where we introduced $F(x,t)$ as

(5.2)

$$\begin{aligned} F(x,t) &= \left[\rho_2 \partial_t^{\frac{1}{2}} - (\rho_1\partial_t - A) \right] (f(x)R(x,t)) + \rho_2 f(x) \frac{R(x,0)}{\sqrt{\pi t}} \\ &= R(x,t) \sum_{i,j=1}^n \partial_i (a_{ij}(x)\partial_j f(x)) \\ &\quad + \sum_{j=1}^n \left(2 \sum_{i=1}^n a_{ij}(x)\partial_i R(x,t) - b_j(x)R(x,t) \right) \partial_j f(x) \\ &\quad + \left[\rho_2 \partial_t^{\frac{1}{2}} R(x,t) - \rho_1 \partial_t R(x,t) + \sum_{i,j=1}^n \partial_i (a_{ij}(x)\partial_j R(x,t)) \right. \\ &\qquad \left. - \sum_{j=1}^n b_j(x)\partial_j R(x,t) - c(x)R(x,t) + \frac{\rho_2 R(x,0)}{\sqrt{\pi t}} \right] f(x), \ (x,t) \in Q. \end{aligned}$$

Let us set $y = \partial_t u$, $z = \partial_t^2 u$ in Q. By differentiating (5.1) with respect to t, we have

$$\rho_2^2 \partial_t y(x,t) - (\rho_1 \partial_t - A)^2 y(x,t) = \partial_t F(x,t), \qquad (x,t) \in Q, \tag{5.3}$$

$$\rho_2^2 \partial_t z(x,t) - (\rho_1 \partial_t - A)^2 z(x,t) = \partial_t^2 F(x,t), \qquad (x,t) \in Q. \tag{5.4}$$

Since $u(x,t) = 0$, $(x,t) \in \partial\Omega \times (0,T)$, we see that

$$y(x,t) = z(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T).$$

To use the Carleman estimate in Q_δ, we introduce the weight functions. Set

$$\varphi_{\delta,1}(x,t) = \frac{e^{\lambda d_1(x)}}{(t - t_0 + \delta)(t_0 + \delta - t)}, \; \psi_{\delta,1}(x,t) = \frac{e^{\lambda d_1(x)} - e^{2\lambda \|d_1\|_{C(\overline{\Omega})}}}{(t - t_0 + \delta)(t_0 + \delta - t)}, \; (x,t) \in Q_\delta.$$

Fixing $\lambda > 0$ and applying Theorem 4.2 ($p = 0$) to (5.3) and (5.4) in Q_δ, we have

$$\begin{aligned}\int_{Q_\delta} \Bigg[(s\varphi_{\delta,1})^3 \left(|\partial_t y|^2 + |\partial_t z|^2 + \sum_{i,j=1}^n |\partial_i \partial_j y|^2 + \sum_{i,j=1}^n |\partial_i \partial_j z|^2 \right) \\ + (s\varphi_{\delta,1})^5 \left(|\nabla y|^2 + |\nabla z|^2 \right) + (s\varphi_{\delta,1})^7 \left(|y|^2 + |z|^2 \right) \Bigg] e^{2s\psi_{\delta,1}} \, dxdt \\ \le C \int_{Q_\delta} s\varphi_{\delta,1} \left(|\partial_t F|^2 + |\partial_t^2 F|^2 \right) e^{2s\psi_{\delta,1}} \, dxdt + C\widetilde{B},\end{aligned} \tag{5.5}$$

where

$$\begin{aligned}\widetilde{B} = \int_{\Sigma_\delta} \Bigg[s\varphi_{\delta,1} \left(|\nabla \partial_t y|^2 + |\nabla \partial_t z|^2 \right) \\ + (s\varphi_{\delta,1})^2 \left(|\nabla \partial_t^{\frac{1}{2}} y|^2 + |\nabla \partial_t^{\frac{1}{2}} z|^2 \right) + (s\varphi_{\delta,1})^5 \left(|\nabla y|^2 + |\nabla z|^2 \right) \Bigg] e^{2s\psi_{\delta,1}} \, dSdt.\end{aligned}$$

We can estimate $\widetilde{B}$ by B^2. We note that $\partial_t^{\frac{1}{2}} \partial_t^m = \partial_t^{m+\frac{1}{2}}$, $m \in \mathbb{N}$, and Lemma A.1. Since there exist constants $C_k(s) > 0$ such that $\varphi_{\delta,1}^k e^{2s\psi_{\delta,1}} \le C_k(s)$ on Σ_δ for $k = 0, 1, 2, \ldots$, we have

$$\begin{aligned}\widetilde{B} \le C \int_{\Sigma_\delta} \Bigg[s\varphi_{\delta,1} |\nabla \partial_t^3 u|^2 + (s\varphi_{\delta,1})^2 \left(|\nabla \partial_t^{\frac{1}{2}} \partial_t u|^2 + |\nabla \partial_t^{\frac{1}{2}} \partial_t^2 u|^2 \right) \\ + (s\varphi_{\delta,1})^5 \left(|\nabla \partial_t u|^2 + |\nabla \partial_t^2 u|^2 \right) \Bigg] e^{2s\psi_{\delta,1}} \, dSdt \\ \le C(s) B^2.\end{aligned}$$

Note that

$$\int_{Q_\delta} s\varphi_{\delta,1} \left(|\partial_t F|^2 + |\partial_t^2 F|^2 \right) e^{2s\psi_{\delta,1}} \, dxdt \le C \int_{Q_\delta} s\varphi_{\delta,1} \sum_{|\alpha| \le 2} |\partial_x^\alpha f|^2 e^{2s\psi_{\delta,1}} \, dxdt.$$

This together with (5.5) gives

$$\int_{Q_\delta}\left[(s\varphi_{\delta,1})^3\left(|\partial_t y|^2+|\partial_t z|^2+\sum_{i,j=1}^n|\partial_i\partial_j y|^2+\sum_{i,j=1}^n|\partial_i\partial_j z|^2\right)\right.$$
$$\left.+(s\varphi_{\delta,1})^5\left(|\nabla y|^2+|\nabla z|^2\right)+(s\varphi_{\delta,1})^7\left(|y|^2+|z|^2\right)\right]e^{2s\psi_{\delta,1}}\,dxdt$$
$$\le C\int_{Q_\delta}s\varphi_{\delta,1}\sum_{|\alpha|\le 2}|\partial_x^\alpha f|^2e^{2s\psi_{\delta,1}}\,dxdt+C(s)B^2. \tag{5.6}$$

Let us expand the left-hand side of (5.1). We have

$$\rho_2^2\partial_t u(x,t)-\rho_1^2\partial_t^2u(x,t)+2\rho_1\partial_t Au(x,t)-A^2u(x,t)=F(x,t),\quad (x,t)\in Q.$$

In particular at $t=t_0$, we have

$$\rho_2^2\partial_t u(x,t_0)-\rho_1^2\partial_t^2u(x,t_0)+2\rho_1\partial_t Au(x,t_0)-A^2u(x,t_0)=F(x,t_0),\quad x\in\Omega. \tag{5.7}$$

Taking the weighted L^2 norm of (5.7) in Ω, we obtain

$$\int_\Omega\varphi_{\delta,1}(x,t_0)|F(x,t_0)|^2e^{2s\psi_{\delta,1}(x,t_0)}\,dx$$
$$\le C\sum_{k=1}^3J_k+C\int_\Omega\varphi_{\delta,1}(x,t_0)\sum_{|\alpha|\le 4}|\partial_x^\alpha u(x,t_0)|^2e^{2s\psi_{\delta,1}(x,t_0)}\,dx, \tag{5.8}$$

where

$$J_1=\int_\Omega\varphi_{\delta,1}(x,t_0)|\partial_t u(x,t_0)|^2e^{2s\psi_{\delta,1}(x,t_0)}\,dx,$$
$$J_2=\int_\Omega\varphi_{\delta,1}(x,t_0)|\partial_t^2 u(x,t_0)|^2e^{2s\psi_{\delta,1}(x,t_0)}\,dx,$$
$$J_3=\int_\Omega\varphi_{\delta,1}(x,t_0)|\partial_t Au(x,t_0)|^2e^{2s\psi_{\delta,1}(x,t_0)}\,dx.$$

Let us estimate J_1, J_2, J_3. We assume that $s>1$ is large enough to satisfy $s\varphi_{\delta,1}>1$ in Q. We note that $\partial_t\psi_{\delta,1}(x,t)\le C\varphi_{\delta,1}^2(x,t)$ for $(x,t)\in Q$.

$$J_1=\int_{t_0-\delta}^{t_0}\int_\Omega\partial_t\left(\varphi_{\delta,1}|y|^2e^{2s\psi_{\delta,1}}\right)\,dxdt$$
$$\le C\int_{t_0-\delta}^{t_0}\int_\Omega\left[\varphi_{\delta,1}^2|y|^2+\varphi_{\delta,1}|\partial_t y||y|+s\varphi_{\delta,1}^3|y|^2\right]e^{2s\psi_{\delta,1}}\,dxdt$$
$$\le C\int_{Q_\delta}s\varphi_{\delta,1}^3\left(|y|^2+|z|^2\right)e^{2s\psi_{\delta,1}}\,dxdt.$$

Combining this with (5.6), we may estimate the right-hand side of the above inequality and we obtain

$$J_1\le\frac{C}{s^5}\int_{Q_\delta}\varphi_{\delta,1}\sum_{|\alpha|\le 2}|\partial_x^\alpha f|^2e^{2s\psi_{\delta,1}}\,dxdt+C(s)B^2. \tag{5.9}$$

Similarly, we obtain

$$
\begin{aligned}
J_2 &= \int_{t_0-\delta}^{t_0} \int_\Omega \partial_t \left(\varphi_{\delta,1} |\partial_t y|^2 e^{2s\psi_{\delta,1}} \right) dxdt \\
&\le C \int_{t_0-\delta}^{t_0} \int_\Omega \left[\varphi_{\delta,1}^2 |\partial_t y|^2 + \varphi_{\delta,1} |\partial_t^2 y||\partial_t y| + s\varphi_{\delta,1}^3 |\partial_t y|^2 \right] e^{2s\psi_{\delta,1}} \, dxdt \\
&\le C \int_{Q_\delta} s\varphi_{\delta,1}^3 \left(|\partial_t y|^2 + |\partial_t z|^2 \right) e^{2s\psi_{\delta,1}} \, dxdt.
\end{aligned}
$$

Putting this together with (5.6), we see that

$$
J_2 \le \frac{C}{s} \int_{Q_\delta} \varphi_{\delta,1} \sum_{|\alpha|\le 2} |\partial_x^\alpha f|^2 e^{2s\psi_{\delta,1}} \, dxdt + C(s)B^2. \tag{5.10}
$$

Moreover we have

$$
\begin{aligned}
J_3 &= \int_{t_0-\delta}^{t_0} \int_\Omega \partial_t \left(\varphi_{\delta,1} |Ay|^2 e^{2s\psi_{\delta,1}} \right) dxdt \\
&\le C \int_{t_0-\delta}^{t_0} \int_\Omega \left[\varphi_{\delta,1}^2 |Ay|^2 + \varphi_{\delta,1} |\partial_t Ay||Ay| + s\varphi_{\delta,1}^3 |Ay|^2 \right] e^{2s\psi_{\delta,1}} \, dxdt \\
&\le C \int_{Q_\delta} s\varphi_{\delta,1}^3 \left(|Ay|^2 + |Az|^2 \right) e^{2s\psi_{\delta,1}} \, dxdt \\
&\le C \int_{Q_\delta} s\varphi_{\delta,1}^3 \left(\sum_{|\alpha|\le 2} |\partial_x^\alpha y|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha z|^2 \right) e^{2s\psi_{\delta,1}} \, dxdt.
\end{aligned}
$$

This together with (5.6) gives

$$
J_3 \le \frac{C}{s} \int_{Q_\delta} \varphi_{\delta,1} \sum_{|\alpha|\le 2} |\partial_x^\alpha f|^2 e^{2s\psi_{\delta,1}} \, dxdt + C(s)B^2. \tag{5.11}
$$

By (5.8) through (5.11), we have

$$
\begin{aligned}
&\int_\Omega \varphi_{\delta,1}(x,t_0) |F(x,t_0)|^2 e^{2s\psi_{\delta,1}(x,t_0)} \, dx \\
&\le \frac{C}{s} \int_{Q_\delta} \varphi_{\delta,1} \sum_{|\alpha|\le 2} |\partial_x^\alpha f|^2 e^{2s\psi_{\delta,1}} \, dxdt \\
&\quad + C \int_\Omega \varphi_{\delta,1}(x,t_0) \sum_{|\alpha|\le 4} |\partial_x^\alpha u(x,t_0)|^2 e^{2s\psi_{\delta,1}(x,t_0)} \, dx + C(s)B^2.
\end{aligned} \tag{5.12}
$$

We will estimate the left-hand side of the inequality (5.12) from below using the Carleman estimate for the elliptic equation stated in Lemma 4.6 ($p = 1$). By (5.2) at

$t = t_0$, we have

$$
\begin{aligned}
(5.13) \qquad & \sum_{i,j=1}^{n} \partial_i(a_{ij}(x)\partial_j \widetilde{f}(x)) \\
& + \frac{1}{R(x,t_0)} \sum_{j=1}^{n} \left(2\sum_{i=1}^{n} a_{ij}(x)\partial_i R(x,t_0) - b_j(x)R(x,t_0) \right) \partial_j \widetilde{f}(x) \\
& + \frac{1}{R(x,t_0)} \Bigg[\rho_2 \partial_t^{\frac{1}{2}} R(x,t_0) - \rho_1 \partial_t R(x,t_0) + \sum_{i,j=1}^{n} \partial_i(a_{ij}(x)\partial_j R(x,t_0)) \\
& \qquad - \sum_{j=1}^{n} b_j(x)\partial_j R(x,t_0) - c(x)R(x,t_0) + \frac{\rho_2 R(x,0)}{\sqrt{\pi t_0}} \Bigg] \widetilde{f}(x) \\
= {} & \frac{F(x,t_0)}{R(x,t_0)}, \quad x \in \Omega.
\end{aligned}
$$

We note that $f(x) = 0$, $x \in \partial\Omega$, and $\nabla f(x) = 0$, $x \in \gamma$, are assumed. Applying Lemma 4.6 to (5.13) in Ω, we obtain

$$
\begin{aligned}
(5.14) \qquad & \frac{1}{s}\int_{\Omega} \varphi_{\delta,1}(x,t_0) \sum_{|\alpha|\le 2} |\partial_x^{\alpha} f(x)|^2 e^{2s\psi_{\delta,1}(x,t_0)}\, dx \\
& \le \frac{C}{s}\int_{\Omega} \sum_{|\alpha|\le 2} |\partial_x^{\alpha} f(x)|^2 e^{2s\psi_{\delta,1}(x,t_0)}\, dx \\
& \le \frac{C}{s}\int_{\Omega} \Bigg(\sum_{i,j=1}^{n} |\partial_i\partial_j f(x)|^2 \\
& \qquad + (s\varphi_{\delta,1}(x,t_0))^2 |\nabla f(x)|^2 + (s\varphi_{\delta,1}(x,t_0))^4 |f(x)|^2 \Bigg) e^{2s\psi_{\delta,1}(x,t_0)}\, dx \\
& \le C\int_{\Omega} \varphi_{\delta,1}(x,t_0) \left| \frac{F(x,t_0)}{R(x,t_0)} \right|^2 e^{2s\psi(x,t_0)}\, dx \\
& \le C\int_{\Omega} \varphi_{\delta,1}(x,t_0) \left| F(x,t_0) \right|^2 e^{2s\psi(x,t_0)}\, dx.
\end{aligned}
$$

By (5.12) and (5.14), we obtain

$$
\begin{aligned}
(5.15) \qquad & \frac{1}{s}\int_{\Omega} \varphi_{\delta,1}(x,t_0) \sum_{|\alpha|\le 2} |\partial_x^{\alpha} f(x)|^2 e^{2s\psi_{\delta,1}(x,t_0)}\, dx \\
& \le \frac{C}{s}\int_{Q_\delta} \varphi_{\delta,1} \sum_{|\alpha|\le 2} |\partial_x^{\alpha} f|^2 e^{2s\psi_{\delta,1}}\, dxdt \\
& \quad + C\int_{\Omega} \varphi_{\delta,1}(x,t_0) \sum_{|\alpha|\le 4} |\partial_x^{\alpha} u(x,t_0)|^2 e^{2s\psi_{\delta,1}(x,t_0)}\, dx + C(s)B^2.
\end{aligned}
$$

Let us estimate the first integral term on the right-hand side of (5.15).

$$
\int_{Q_\delta} \varphi_{\delta,1} \sum_{|\alpha|\le 2} |\partial_x^{\alpha} f|^2 e^{2s\psi_{\delta,1}}\, dxdt = \int_{\Omega} \varphi_{\delta,1}(x,t_0) \sum_{|\alpha|\le 2} |\partial_x^{\alpha} f|^2 e^{2s\psi_{\delta,1}(x,t_0)} h_s(x)\, dx,
$$

where

$$h_s(x) = \frac{1}{\varphi_{\delta,1}(x,t_0)} \int_{t_0-\delta}^{t_0+\delta} \varphi_{\delta,1} e^{-2s(\psi_{\delta,1}(x,t_0)-\psi_{\delta,1}(x,t))}\, dt, \quad x \in \Omega.$$

Since $\psi_{\delta,1}(x,t_0) - \psi_{\delta,1}(x,t) \geq 0$, $(x,t) \in Q_\delta$, h_s converges pointwise to 0 in Ω as $s \to \infty$ by Lebesgue's dominated convergence theorem. Moreover by Dini's theorem, we see that h_s converges uniformly to 0 in Ω as $s \to \infty$. Hence, taking sufficiently large $s > 0$, we can absorb the first term on the right-hand side of (5.15) into the left-hand side and obtain

$$\begin{aligned}(5.16)\qquad &\frac{1}{s}\int_\Omega \varphi_{\delta,1}(x,t_0) \sum_{|\alpha|\leq 2} |\partial_x^\alpha f(x)|^2 e^{2s\psi_{\delta,1}(x,t_0)}\, dx \\ &\leq C \int_\Omega \varphi_{\delta,1}(x,t_0) \sum_{|\alpha|\leq 4} |\partial_x^\alpha u(x,t_0)|^2 e^{2s\psi_{\delta,1}(x,t_0)}\, dx + C(s)B^2.\end{aligned}$$

Fix $s > 0$. Noting that $\varphi_{\delta,1}(\cdot,t_0)e^{2s\psi_{\delta,1}(\cdot,t_0)}$ has its upper and lower bounds in $\overline{\Omega}$, we see that

$$\|f\|_{H^2(\Omega)} \leq C\|u(\cdot,t_0)\|_{H^4(\Omega)} + CB.$$

Thus we obtain the stability estimate. ∎

Proof of Theorem 2.2. We may prove Theorem 2.2 by an argument similar to that used in the proof of Theorem 2.1. In the proof, Theorem 4.3 and Lemma 4.7 are used instead of Theorem 4.2 and Lemma 4.6. ∎

5.2. Stability for the diffusion coefficient. Next we prove Theorems 3.5 and 3.6. The proofs are very similar to the proofs of Theorems 2.1 and 2.2.

Proof of Theorem 3.5. Applying Lemma 4.1 to (3.7), we obtain

$$(5.17)\qquad \rho_2^2 \partial_t u(x,t) - (\rho_1\partial_t - A_2^{(1)})^2 u(x,t) = \widetilde{F}(x,t), \quad (x,t) \in Q,$$

where

(5.18)

$$\begin{aligned}\widetilde{F}(x,t) &= \left[\rho_2\partial_t^{\frac{1}{2}} - (\rho_1\partial_t - A_2^{(1)})\right](\operatorname{div}(a(x)\nabla r(x,t))) + \frac{\rho_2 \operatorname{div}(a(x)\nabla r(x,0))}{\sqrt{\pi t}} \\ &= a^{(1)}(x)\nabla r(x,t)\cdot\nabla\triangle a(x) + 2a^{(1)}(x)\sum_{i,j=1}^n (\partial_i\partial_j r(x,t))(\partial_i\partial_j a(x)) \\ &\quad + a^{(1)}(x)\triangle r(x,t)\triangle a(x) + (\nabla a^{(1)}(x) - \mathbf{b}(x))\cdot(\nabla r(x,t)\cdot\nabla)\nabla a(x) \\ &\quad + \Bigg[(\rho_2\partial_t^{\frac{1}{2}} - \rho_1\partial_t)\nabla r(x,t) + 3a^{(1)}(x)\nabla\triangle r(x,t) + (\triangle r(x,t))\,\nabla a^{(1)}(x) \\ &\qquad - (\triangle r(x,t))\,\mathbf{b}(x) - c(x)\nabla r(x,t) + \frac{\rho_2\nabla r(x,0)}{\sqrt{\pi t}}\Bigg]\cdot\nabla a(x) \\ &\quad + (\nabla a^{(1)}(x) - \mathbf{b}(x))\cdot(\nabla a(x)\cdot\nabla)\nabla r(x,t) \\ &\quad + \Bigg[(\rho_2\partial_t^{\frac{1}{2}} - \rho_1\partial_t)\triangle r(x,t) + \Big(\nabla a^{(1)}(x)\cdot\nabla\triangle r(x,t)\Big) + a^{(1)}(x)\triangle^2 r(x,t) \\ &\qquad - (\mathbf{b}(x)\cdot\nabla\triangle r(x,t)) - c(x)\triangle r(x,t) + \frac{\rho_2\triangle r(x,0)}{\sqrt{\pi t}}\Bigg]a(x), \quad (x,t) \in Q.\end{aligned}$$

Setting $y = \partial_t u$, $z = \partial_t^2 u$ in Q and differentiating (5.17) with respect to t, we have

$$\rho_2^2 \partial_t y(x,t) - (\rho_1 \partial_t - A_2^{(1)})^2 y(x,t) = \partial_t \widetilde{F}(x,t), \qquad (x,t) \in Q, \tag{5.19}$$

$$\rho_2^2 \partial_t z(x,t) - (\rho_1 \partial_t - A_2^{(1)})^2 z(x,t) = \partial_t^2 \widetilde{F}(x,t), \qquad (x,t) \in Q. \tag{5.20}$$

Since $u(x,t) = 0$, $(x,t) \in \partial\Omega \times (0,T)$, we see that

$$y(x,t) = z(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T).$$

Fixing $\lambda > 0$ and applying Theorem 4.2 ($p = 1$) to (5.19) and (5.20) in Q_δ, we have

$$\begin{aligned}
&\int_{Q_\delta} \Bigg[(s\varphi_{\delta,1})^2 \left(|\nabla \partial_t y|^2 + |\nabla \partial_t z|^2 \right) \\
&\quad + (s\varphi_{\delta,1})^3 \left(|\nabla(\rho_1 \partial_t - A_2^{(1)}) y|^2 + |\nabla(\rho_1 \partial_t - A_2^{(1)}) z|^2 \right) \\
&\quad + (s\varphi_{\delta,1})^4 \left(|\partial_t y|^2 + |\partial_t z|^2 + \sum_{i,j=1}^n |\partial_i \partial_j y|^2 + \sum_{i,j=1}^n |\partial_i \partial_j z|^2 \right) \\
&\quad + (s\varphi_{\delta,1})^6 \left(|\nabla y|^2 + |\nabla z|^2 \right) + (s\varphi_{\delta,1})^8 \left(|y|^2 + |z|^2 \right) \Bigg] e^{2s\psi_{\delta,1}}\, dxdt \\
&\le C \int_{Q_\delta} (s\varphi_{\delta,1})^2 \left(|\partial_t \widetilde{F}|^2 + |\partial_t^2 \widetilde{F}|^2 \right) e^{2s\psi_{\delta,1}}\, dxdt + C\widehat{B},
\end{aligned} \tag{5.21}$$

where

$$\begin{aligned}
\widehat{B} = &\int_{\Sigma_\delta} \Bigg[(s\varphi_{\delta,1})^2 \left(|\nabla \partial_t y|^2 + |\nabla \partial_t z|^2 \right) \\
&+ (s\varphi_{\delta,1})^3 \left(|\nabla \partial_t^{\frac{1}{2}} y|^2 + |\nabla \partial_t^{\frac{1}{2}} z|^2 \right) + (s\varphi_{\delta,1})^6 \left(|\nabla y|^2 + |\nabla z|^2 \right) \Bigg] e^{2s\psi_{\delta,1}}\, dSdt.
\end{aligned}$$

As we have seen in the proof of Theorem 2.1, we may obtain $\widehat{B} \le C(s) B^2$.

Note that

$$\begin{aligned}
&\int_{Q_\delta} (s\varphi_{\delta,1})^2 \left(|\partial_t \widetilde{F}|^2 + |\partial_t^2 \widetilde{F}|^2 \right) e^{2s\psi_{\delta,1}}\, dxdt \\
&\le C \int_{Q_\delta} (s\varphi_{\delta,1})^2 \left(|\nabla \triangle a|^2 + \sum_{|\alpha| \le 2} |\partial_x^\alpha a|^2 \right) e^{2s\psi_{\delta,1}}\, dxdt.
\end{aligned}$$

This together with (5.21) gives

$$\begin{aligned}
&\int_{Q_\delta} \Bigg[(s\varphi_{\delta,1})^2 \left(|\nabla \partial_t y|^2 + |\nabla \partial_t z|^2 \right) \\
&\quad + (s\varphi_{\delta,1})^3 \left(|\nabla(\rho_1 \partial_t - A_2^{(1)}) y|^2 + |\nabla(\rho_1 \partial_t - A_2^{(1)}) z|^2 \right)
\end{aligned} \tag{5.22}$$

$$+ (s\varphi_{\delta,1})^4 \left(|\partial_t y|^2 + |\partial_t z|^2 + \sum_{i,j=1}^n |\partial_i \partial_j y|^2 + \sum_{i,j=1}^n |\partial_i \partial_j z|^2 \right)$$
$$+ (s\varphi_{\delta,1})^6 \left(|\nabla y|^2 + |\nabla z|^2 \right) + (s\varphi_{\delta,1})^8 \left(|y|^2 + |z|^2 \right) \Bigg] e^{2s\psi_{\delta,1}}\, dxdt$$
$$\le C \int_{Q_\delta} (s\varphi_{\delta,1})^2 \left(|\nabla \triangle a|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha a|^2 \right) e^{2s\psi_{\delta,1}}\, dxdt + C(s)B^2.$$

Let us expand the left-hand side of (5.17). We have

$$\rho_2^2 \partial_t u(x,t) - \rho_1^2 \partial_t^2 u(x,t) + 2\rho_1 \partial_t A_2^{(1)} u(x,t) - (A_2^{(1)})^2 u(x,t) = \widetilde{F}(x,t), \quad (x,t) \in Q.$$

Moreover we have

$$\rho_2^2 \nabla \partial_t u(x,t) - \rho_1^2 \nabla \partial_t^2 u(x,t) + 2\rho_1 \nabla \partial_t A_2^{(1)} u(x,t) - \nabla (A_2^{(1)})^2 u(x,t) = \nabla \widetilde{F}(x,t), (x,t) \in Q.$$

In particular at $t = t_0$, we have

$$\rho_2^2 \partial_t u(x,t_0) - \rho_1^2 \partial_t^2 u(x,t_0) + 2\rho_1 \partial_t A_2^{(1)} u(x,t_0) - (A_2^{(1)})^2 u(x,t_0) = \widetilde{F}(x,t_0), \ x \in \Omega, \tag{5.23}$$

and

$$\rho_2^2 \nabla \partial_t u(x,t_0) - \rho_1^2 \nabla \partial_t^2 u(x,t_0) + 2\rho_1 \nabla \partial_t A_2^{(1)} u(x,t_0) - \nabla (A_2^{(1)})^2 u(x,t_0) = \nabla \widetilde{F}(x,t_0), \quad x \in \Omega. \tag{5.24}$$

Taking the weighted L^2 norm of (5.23) and (5.24) in Ω, we obtain

$$\int_\Omega \left(|\widetilde{F}(x,t_0)|^2 + |\nabla \widetilde{F}(x,t_0)|^2 \right) e^{2s\psi_{\delta,1}(x,t_0)}\, dx \le C \sum_{k=1}^6 \widetilde{J}_k + C \int_\Omega \sum_{|\alpha|\le 5} |\partial_x^\alpha u(x,t_0)|^2 e^{2s\psi_{\delta,1}(x,t_0)}\, dx, \tag{5.25}$$

where

$$\widetilde{J}_1 = \int_\Omega |\partial_t u(x,t_0)|^2 e^{2s\psi_{\delta,1}(x,t_0)}\, dx, \qquad \widetilde{J}_2 = \int_\Omega |\partial_t^2 u(x,t_0)|^2 e^{2s\psi_{\delta,1}(x,t_0)}\, dx,$$
$$\widetilde{J}_3 = \int_\Omega |\partial_t A_2^{(1)} u(x,t_0)|^2 e^{2s\psi_{\delta,1}(x,t_0)}\, dx, \quad \widetilde{J}_4 = \int_\Omega |\nabla \partial_t u(x,t_0)|^2 e^{2s\psi_{\delta,1}(x,t_0)}\, dx,$$
$$\widetilde{J}_5 = \int_\Omega |\nabla \partial_t^2 u(x,t_0)|^2 e^{2s\psi_{\delta,1}(x,t_0)}\, dx, \qquad \widetilde{J}_6 = \int_\Omega |\nabla \partial_t A_2^{(1)} u(x,t_0)|^2 e^{2s\psi_{\delta,1}(x,t_0)}\, dx.$$

Henceforth we estimate $\widetilde{J}_1$ through $\widetilde{J}_6$ by using the Carleman estimate. We assume that $s > 1$ is large enough to satisfy $s\varphi_{\delta,1} > 1$ in Q. We note that $\partial_t \psi_{\delta,1}(x,t) \le$

$C\varphi_{\delta,1}^2(x,t)$ for $(x,t)\in Q$.

$$\begin{aligned}\widetilde{J}_1 &= \int_{t_0-\delta}^{t_0}\int_\Omega \partial_t\left(|y|^2 e^{2s\psi_{\delta,1}}\right)\,dxdt\\ &\le C\int_{t_0-\delta}^{t_0}\int_\Omega \left(|\partial_t y||y| + s\varphi_{\delta,1}^2|y|^2\right)e^{2s\psi_{\delta,1}}\,dxdt\\ &\le C\int_{Q_\delta} s\varphi_{\delta,1}^2\left(|y|^2+|z|^2\right)e^{2s\psi_{\delta,1}}\,dxdt.\end{aligned}$$

Combining this with (5.22), we may estimate the right-hand side of the above inequality and we obtain

$$\widetilde{J}_1 \le \frac{C}{s^5}\int_{Q_\delta}\varphi_{\delta,1}^2\left(|\nabla\triangle a|^2 + \sum_{|\alpha|\le 2}|\partial_x^\alpha a|^2\right)e^{2s\psi_{\delta,1}}\,dxdt + C(s)B^2. \tag{5.26}$$

Similarly, we obtain

$$\begin{aligned}\widetilde{J}_2 &= \int_{t_0-\delta}^{t_0}\int_\Omega \partial_t\left(|\partial_t y|^2 e^{2s\psi_{\delta,1}}\right)\,dxdt\\ &\le C\int_{t_0-\delta}^{t_0}\int_\Omega \left(|\partial_t^2 y||\partial_t y| + s\varphi_{\delta,1}^2|\partial_t y|^2\right)e^{2s\psi_{\delta,1}}\,dxdt\\ &\le C\int_{Q_\delta} s\varphi_{\delta,1}^2\left(|\partial_t y|^2+|\partial_t z|^2\right)e^{2s\psi_{\delta,1}}\,dxdt.\end{aligned}$$

Putting this together with (5.22), we see that

$$\widetilde{J}_2 \le \frac{C}{s}\int_{Q_\delta}\varphi_{\delta,1}^2\left(|\nabla\triangle a|^2 + \sum_{|\alpha|\le 2}|\partial_x^\alpha a|^2\right)e^{2s\psi_{\delta,1}}\,dxdt + C(s)B^2. \tag{5.27}$$

Moreover we have

$$\begin{aligned}\widetilde{J}_3 &= \int_{t_0-\delta}^{t_0}\int_\Omega \partial_t\left(|A_2^{(1)}y|^2 e^{2s\psi_{\delta,1}}\right)\,dxdt\\ &\le C\int_{t_0-\delta}^{t_0}\int_\Omega \left(|A_2^{(1)}\partial_t y||A_2^{(1)}y| + s\varphi_{\delta,1}^2|A_2^{(1)}y|^2\right)e^{2s\psi_{\delta,1}}\,dxdt\\ &\le C\int_{Q_\delta} s\varphi_{\delta,1}^2\left(|A_2^{(1)}y|^2+|A_2^{(1)}z|^2\right)e^{2s\psi_{\delta,1}}\,dxdt\\ &\le C\int_{Q_\delta} s\varphi_{\delta,1}^2\left(\sum_{|\alpha|\le 2}|\partial_x^\alpha y|^2+\sum_{|\alpha|\le 2}|\partial_x^\alpha z|^2\right)e^{2s\psi_{\delta,1}}\,dxdt.\end{aligned}$$

This together with (5.22) gives

$$\widetilde{J}_3 \le \frac{C}{s}\int_{Q_\delta}\varphi_{\delta,1}^2\left(|\nabla\triangle a|^2 + \sum_{|\alpha|\le 2}|\partial_x^\alpha a|^2\right)e^{2s\psi_{\delta,1}}\,dxdt + C(s)B^2. \tag{5.28}$$

We have

$$\begin{aligned}\widetilde{J}_4 &= \int_{t_0-\delta}^{t_0}\int_\Omega \partial_t\left(|\nabla y|^2 e^{2s\psi_{\delta,1}}\right)\,dxdt\\ &\le C\int_{t_0-\delta}^{t_0}\int_\Omega \left(|\nabla\partial_t y||\nabla y| + s\varphi_{\delta,1}^2|\nabla y|^2\right)e^{2s\psi_{\delta,1}}\,dxdt\\ &\le C\int_{Q_\delta} s\varphi_{\delta,1}^2\left(|\nabla y|^2+|\nabla z|^2\right)e^{2s\psi_{\delta,1}}\,dxdt.\end{aligned}$$

Combining this with (5.22), we may estimate the right-hand side of the above inequality and we obtain

$$\widetilde{J}_4 \le \frac{C}{s^3}\int_{Q_\delta}\varphi_{\delta,1}^2\left(|\nabla\triangle a|^2 + \sum_{|\alpha|\le 2}|\partial_x^\alpha a|^2\right)e^{2s\psi_{\delta,1}}\,dxdt + C(s)B^2. \tag{5.29}$$

Similarly, we obtain

$$\begin{aligned}\widetilde{J}_5 &= \int_{t_0-\delta}^{t_0}\int_\Omega \partial_t\left(|\nabla\partial_t y|^2 e^{2s\psi_{\delta,1}}\right)\,dxdt\\ &\le C\int_{t_0-\delta}^{t_0}\int_\Omega \left(|\nabla\partial_t^2 y||\nabla\partial_t y| + s\varphi_{\delta,1}^2|\nabla\partial_t y|^2\right)e^{2s\psi_{\delta,1}}\,dxdt\\ &\le C\int_{Q_\delta} s\varphi_{\delta,1}^2\left(|\nabla\partial_t y|^2+|\nabla\partial_t z|^2\right)e^{2s\psi_{\delta,1}}\,dxdt.\end{aligned}$$

Putting this together with (5.22), we see that

$$\widetilde{J}_5 \le C\int_{Q_\delta}s\varphi_{\delta,1}^2\left(|\nabla\triangle a|^2 + \sum_{|\alpha|\le 2}|\partial_x^\alpha a|^2\right)e^{2s\psi_{\delta,1}}\,dxdt + C(s)B^2. \tag{5.30}$$

Moreover we have

$$\begin{aligned}\widetilde{J}_6 &= \int_{t_0-\delta}^{t_0}\int_\Omega \partial_t\left(|\nabla A_2^{(1)}y|^2 e^{2s\psi_{\delta,1}}\right)\,dxdt\\ &\le C\int_{t_0-\delta}^{t_0}\int_\Omega \left(|\nabla A_2^{(1)}\partial_t y||\nabla \mathcal{A}_1 y| + s\varphi_{\delta,1}^2|\nabla A_2^{(1)}y|^2\right)e^{2s\psi_{\delta,1}}\,dxdt\\ &\le C\int_{Q_\delta} s\varphi_{\delta,1}^2\left(|\nabla A_2^{(1)}y|^2+|\nabla A_2^{(1)}z|^2\right)e^{2s\psi_{\delta,1}}\,dxdt\\ &\le C\int_{Q_\delta} s\varphi_{\delta,1}^2\Big(|\nabla\partial_t y|^2 + |\nabla(\rho_1\partial_t - A_2^{(1)})y|^2\\ &\qquad + |\nabla\partial_t z|^2 + |\nabla(\rho_1\partial_t - A_2^{(1)})z|^2\Big)e^{2s\psi_{\delta,1}}\,dxdt.\end{aligned}$$

This together with (5.22) gives

$$\widetilde{J}_6 \le C\int_{Q_\delta}s\varphi_{\delta,1}^2\left(|\nabla\triangle a|^2 + \sum_{|\alpha|\le 2}|\partial_x^\alpha a|^2\right)e^{2s\psi_{\delta,1}}\,dxdt + C(s)B^2. \tag{5.31}$$

Summing up the estimate of (5.25) through (5.31), we have

$$
\begin{aligned}
&\int_\Omega \left(|\widetilde{F}(x,t_0)|^2 + |\nabla \widetilde{F}(x,t_0)|^2\right) e^{2s\psi_{\delta,1}(x,t_0)}\,dx \\
&\le C\int_{Q_\delta} s\varphi_{\delta,1}^2 \left(|\nabla\triangle a|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha a|^2\right) e^{2s\psi_{\delta,1}}\,dxdt \\
&\quad + C\int_\Omega \sum_{|\alpha|\le 5} |\partial_x^\alpha u(x,t_0)|^2 e^{2s\psi_{\delta,1}(x,t_0)}\,dx + C(s)B^2.
\end{aligned}
\tag{5.32}
$$

Let us estimate the left-hand side of the inequality (5.32) from below. By (5.18) at $t = t_0$, we have

$$
\begin{aligned}
&a^{(1)}(x)\nabla r(x,t_0)\cdot\nabla\triangle a(x) \\
&= \widetilde{F}(x,t_0) - 2a^{(1)}(x)\sum_{i,j=1}^n (\partial_i\partial_j r(x,t_0))(\partial_i\partial_j a(x)) \\
&\quad - a^{(1)}(x)\triangle r(x,t_0)\triangle a(x) - (\nabla a^{(1)}(x) - \mathbf{b}(x))\cdot(\nabla r(x,t_0)\cdot\nabla)\nabla a(x) \\
&\quad - \Bigg[(\rho_2\partial_t^{\frac12} - \rho_1\partial_t)\nabla r(x,t_0) + 3a^{(1)}(x)\nabla\triangle r(x,t_0) + (\triangle r(x,t_0))\,\nabla a^{(1)}(x) \\
&\qquad - (\triangle r(x,t_0))\,\mathbf{b}(x) - c(x)\nabla r(x,t_0) + \frac{\rho_2\nabla r(x,0)}{\sqrt{\pi t_0}}\Bigg]\cdot\nabla a(x) \\
&\quad - (\nabla a^{(1)}(x) - \mathbf{b}(x))\cdot(\nabla a(x)\cdot\nabla)\nabla r(x,t_0) \\
&\quad - \Bigg[(\rho_2\partial_t^{\frac12} - \rho_1\partial_t)\triangle r(x,t_0) + \left(\nabla a^{(1)}(x)\cdot\nabla\triangle r(x,t_0)\right) + a^{(1)}(x)\triangle^2 r(x,t_0) \\
&\qquad - (\mathbf{b}(x)\cdot\nabla\triangle r(x,t_0)) - c(x)\triangle r(x,t_0) + \frac{\rho_2\triangle r(x,0)}{\sqrt{\pi t_0}}\Bigg]a(x), \quad x\in\Omega.
\end{aligned}
\tag{5.33}
$$

Note that

$$
|\nabla r(x,t_0)\cdot\nabla d_1(x)| \ge m_1 > 0, \quad x\in\overline{\Omega},
$$

and $a \in H^4(\Omega)$ satisfies $a(x) = 0$, $x \in D$. Let us apply Lemma 4.8 to (5.33) in Ω. Then we obtain

$$
\begin{aligned}
&\int_\Omega \Bigg[s\varphi_{\delta,1}(x,t_0)\sum_{i,j,k=1}^n |\partial_i\partial_j\partial_k a(x)|^2 + (s\varphi_{\delta,1}(x,t_0))^2|\nabla\triangle a(x)|^2 \\
&\quad + (s\varphi_{\delta,1}(x,t_0))^3\sum_{i,j=1}^n |\partial_i\partial_j a(x)|^2 \\
&\quad + (s\varphi_{\delta,1}(x,t_0))^5\left(|\nabla a(x)|^2 + |a(x)|^2\right)\Bigg] e^{2s\psi_{\delta,1}(x,t_0)}\,dx \\
&\le C\int_\Omega \left(|\widetilde{F}(x,t_0)|^2 + |\nabla\widetilde{F}(x,t_0)|^2\right) e^{2s\psi_{\delta,1}(x,t_0)}\,dx + C\int_\Omega \sum_{|\alpha|\le 3} |\partial_x^\alpha a(x)|^2 e^{2s\psi_{\delta,1}(x,t_0)}\,dx.
\end{aligned}
$$

Combining this with (5.32), we obtain

(5.34)
$$\begin{aligned}
&\int_\Omega \Bigg[s\varphi_{\delta,1}(x,t_0) \sum_{i,j,k=1}^n |\partial_i\partial_j\partial_k a(x)|^2 + (s\varphi_{\delta,1}(x,t_0))^2 |\nabla\triangle a(x)|^2 \\
&\qquad + (s\varphi_{\delta,1}(x,t_0))^3 \sum_{i,j=1}^n |\partial_i\partial_j a(x)|^2 \\
&\qquad + (s\varphi_{\delta,1}(x,t_0))^5 \left(|\nabla a(x)|^2 + |a(x)|^2\right) \Bigg] e^{2s\psi_{\delta,1}(x,t_0)}\, dx \\
&\le C \int_{Q_\delta} s\varphi_{\delta,1}^2 \Big(|\nabla\triangle a|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha a|^2 \Big) e^{2s\psi_{\delta,1}}\, dxdt \\
&\quad + C\int_\Omega \sum_{|\alpha|\le 3} |\partial_x^\alpha a(x)|^2 e^{2s\psi_{\delta,1}(x,t_0)}\, dx \\
&\quad + C\int_\Omega \sum_{|\alpha|\le 5} |\partial_x^\alpha u(x,t_0)|^2 e^{2s\psi_{\delta,1}(x,t_0)}\, dx + C(s)B^2. \\
&\le C\int_\Omega \int_{Q_\delta} \Bigg[\sum_{i,j,k=1}^n |\partial_i\partial_j\partial_k a(x)|^2 \\
&\qquad\qquad + s\varphi_{\delta,1}^2(x,t_0) \Big(|\nabla\triangle a(x)|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha a(x)|^2 \Big) \Bigg] e^{2s\psi_{\delta,1}(x,t_0)}\, dx \\
&\quad + C\int_\Omega \sum_{|\alpha|\le 5} |\partial_x^\alpha u(x,t_0)|^2 e^{2s\psi_{\delta,1}(x,t_0)}\, dx + C(s)B^2.
\end{aligned}$$

In the last inequality, we used the fact that

$$\varphi_{\delta,1}^2(x,t) e^{2s\psi_{\delta,1}(x,t)} \le \varphi_{\delta,1}^2(x,t_0) e^{2s\psi_{\delta,1}(x,t_0)}, \quad (x,t) \in Q_\delta,$$

for large $s > 0$.

Choose sufficiently large $s > 0$ and absorb the first term on the right-hand side of (5.34) into the left-hand side. Then we obtain

$$\begin{aligned}
&\int_\Omega \Bigg[s\varphi_{\delta,1}(x,t_0) \sum_{i,j,k=1}^n |\partial_i\partial_j\partial_k a(x)|^2 + (s\varphi_{\delta,1}(x,t_0))^2 |\nabla\triangle a(x)|^2 \\
&\qquad + (s\varphi_{\delta,1}(x,t_0))^3 \sum_{i,j=1}^n |\partial_i\partial_j a(x)|^2 \\
&\qquad + (s\varphi_{\delta,1}(x,t_0))^5 \left(|\nabla a(x)|^2 + |a(x)|^2\right) \Bigg] e^{2s\psi_{\delta,1}(x,t_0)}\, dx \\
&\le C\int_\Omega \sum_{|\alpha|\le 5} |\partial_x^\alpha u(x,t_0)|^2 e^{2s\psi_{\delta,1}(x,t_0)}\, dx + C(s)B^2.
\end{aligned} \tag{5.35}$$

Fix $s > 0$. Noting that $\varphi_{\delta,1}(\cdot,t_0)e^{2s\psi_{\delta,1}(\cdot,t_0)}$ has its upper and lower bounds in $\overline{\Omega}$, we see that

$$\|a\|_{H^3(\Omega)} \le C\|u(\cdot,t_0)\|_{H^5(\Omega)} + CB.$$

Thus we obtain the stability estimate (3.11). ∎

Proof of Theorem 3.6. We may prove Theorem 3.6 in the same way as Theorem 3.5. ∎

5.3. Stability in simultaneous determination. Next we prove Theorems 3.8 and 3.9 by the combination of the proofs of Theorems 3.5 and 3.6 and the proofs of Theorems 2.1 and 2.2.

Proof of Theorem 3.8. Applying Lemma 4.1 to (3.17), we obtain

$$\rho_2^2\partial_t u_\ell(x,t) - (\rho_1\partial_t - \mathcal{A}^{(1)})^2 u_\ell(x,t) = F_\ell(x,t), \quad (x,t)\in Q, \tag{5.36}$$

where

(5.37)

$$\begin{aligned}
F_\ell(x,t) &= \left[\rho_2\partial_t^{\frac12} - (\rho_1\partial_t - \mathcal{A}^{(1)})\right](\mathrm{div}(a(x)\nabla r_\ell(x,t)) - c(x)r_\ell(x,t))\\
&\quad + \frac{\rho_2\,\mathrm{div}(a(x)\nabla r_\ell(x,0))}{\sqrt{\pi t}} - \frac{\rho_2 c(x) r_\ell(x,0)}{\sqrt{\pi t}}\\
&= a^{(1)}(x)\nabla r_\ell(x,t)\cdot\nabla\triangle a(x) + 2a^{(1)}(x)\sum_{i,j=1}^n(\partial_i\partial_j r_\ell(x,t))(\partial_i\partial_j a(x))\\
&\quad + a^{(1)}(x)\triangle r_\ell(x,t)\triangle a(x) + \nabla a^{(1)}(x)\cdot(\nabla r_\ell(x,t)\cdot\nabla)\nabla a(x)\\
&\quad + \Bigg[(\rho_2\partial_t^{\frac12} - \rho_1\partial_t)\nabla r_\ell(x,t) + 3a^{(1)}(x)\nabla\triangle r_\ell(x,t) + (\triangle r_\ell(x,t))\nabla a^{(1)}(x)\\
&\qquad - c^{(1)}(x)\nabla r_\ell(x,t) + \frac{\rho_2\nabla r_\ell(x,0)}{\sqrt{\pi t}}\Bigg]\cdot\nabla a(x)\\
&\quad + \nabla a^{(1)}(x)\cdot(\nabla a(x)\cdot\nabla)\nabla r_\ell(x,t)\\
&\quad + \Bigg[(\rho_2\partial_t^{\frac12} - \rho_1\partial_t)\triangle r_\ell(x,t) + \Big(\nabla a^{(1)}(x)\cdot\nabla\triangle r_\ell(x,t)\Big) + a^{(1)}(x)\triangle^2 r_\ell(x,t)\\
&\qquad - c^{(1)}(x)\triangle r_\ell(x,t) + \frac{\rho_2\triangle r_\ell(x,0)}{\sqrt{\pi t}}\Bigg]a(x)\\
&\quad - r_\ell(x,t)\,\mathrm{div}(a^{(1)}(x))\nabla c(x)) - 2a^{(1)}(x)\nabla r_\ell(x,t)\cdot\nabla c(x)\\
&\quad - \Bigg[(\rho_2\partial_t^{\frac12} - \rho_1\partial_t)r_\ell(x,t) + \mathrm{div}(a^{(1)}(x)\nabla r_\ell(x,t))\\
&\qquad - c^{(1)}(x)r_\ell(x,t) + \frac{\rho_2 r_\ell(x,0)}{\sqrt{\pi t}}\Bigg]c(x), \quad (x,t)\in Q,
\end{aligned}$$

for $\ell = 1,2$.

Setting $y_\ell = \partial_t u_\ell$, $z_\ell = \partial_t^2 u_\ell$ in Q for $\ell = 1,2$ and differentiating (5.36) with respect to t, we have

$$\rho_2^2\partial_t y_\ell(x,t) - (\rho_1\partial_t - \mathcal{A}^{(1)})^2 y_\ell(x,t) = \partial_t F_\ell(x,t), \qquad (x,t)\in Q, \tag{5.38}$$

$$\rho_2^2\partial_t z_\ell(x,t) - (\rho_1\partial_t - \mathcal{A}^{(1)})^2 z_\ell(x,t) = \partial_t^2 F_\ell(x,t), \qquad (x,t)\in Q, \tag{5.39}$$

for $\ell = 1,2$. Since $u_\ell(x,t) = 0$, $(x,t) \in \partial\Omega \times (0,T)$, we see that

$$y_\ell(x,t) = z_\ell(x,t) = 0, \quad (x,t)\in\partial\Omega\times(0,T),$$

for $\ell = 1,2$.

Fixing $\lambda > 0$ and applying Theorem 4.2 ($p = 1$) to (5.38) and (5.39) in Q_δ, we have

$$\begin{aligned}\int_{Q_\delta}\Bigg[&(s\varphi_{\delta,1})^2\left(|\nabla\partial_t y_\ell|^2 + |\nabla\partial_t z_\ell|^2\right)\\ &+ (s\varphi_{\delta,1})^3\left(|\nabla(\rho_1\partial_t - \mathcal{A}^{(1)})y_\ell|^2 + |\nabla(\rho_1\partial_t - \mathcal{A}^{(1)})z_\ell|^2\right)\\ &+ (s\varphi_{\delta,1})^4\left(|\partial_t y_\ell|^2 + |\partial_t z_\ell|^2 + \sum_{i,j=1}^n |\partial_i\partial_j y_\ell|^2 + \sum_{i,j=1}^n|\partial_i\partial_j z_\ell|^2\right)\\ &+ (s\varphi_{\delta,1})^6\left(|\nabla y_\ell|^2 + |\nabla z_\ell|^2\right) + (s\varphi_{\delta,1})^8\left(|y_\ell|^2 + |z_\ell|^2\right)\Bigg]e^{2s\psi_{\delta,1}}\,dxdt\\ \le C\int_{Q_\delta}&(s\varphi_{\delta,1})^2\left(|\partial_t F_\ell|^2 + |\partial_t^2 F_\ell|^2\right)e^{2s\psi_{\delta,1}}\,dxdt + C\widehat{B}_\ell,\end{aligned} \tag{5.40}$$

where

$$\begin{aligned}\widehat{B}_\ell = \int_{\Sigma_\delta}\Bigg[&(s\varphi_{\delta,1})^2\left(|\nabla\partial_t y_\ell|^2 + |\nabla\partial_t z_\ell|^2\right) + (s\varphi_{\delta,1})^3\left(|\nabla\partial_t^{\frac12} y_\ell|^2 + |\nabla\partial_t^{\frac12} z_\ell|^2\right)\\ &+ (s\varphi_{\delta,1})^6\left(|\nabla y_\ell|^2 + |\nabla z_\ell|^2\right)\Bigg]e^{2s\psi_{\delta,1}}\,dSdt,\end{aligned}$$

for $\ell = 1,2$. As we have seen in the proof of Theorem 2.1, we may obtain $\widehat{B}_1 + \widehat{B}_2 \le C(s)B^2$.

Note that

$$\begin{aligned}&\int_{Q_\delta}(s\varphi_{\delta,1})^2\left(|\partial_t F_\ell|^2 + |\partial_t^2 F_\ell|^2\right)e^{2s\psi_{\delta,1}}\,dxdt\\ &\le C\int_{Q_\delta}(s\varphi_{\delta,1})^2\left(|\nabla\triangle a|^2 + \sum_{|\alpha|\le 2}|\partial_x^\alpha a|^2 + \sum_{|\alpha|\le 2}|\partial_x^\alpha c|^2\right)e^{2s\psi_{\delta,1}}\,dxdt\end{aligned}$$

for $\ell = 1,2$. This together with (5.40) gives

(5.41)

$$\begin{aligned}\int_{Q_\delta}\Bigg[&(s\varphi_{\delta,1})^2\left(|\nabla\partial_t y_\ell|^2 + |\nabla\partial_t z_\ell|^2\right)\\ &+ (s\varphi_{\delta,1})^3\left(|\nabla(\rho_1\partial_t - \mathcal{A}^{(1)})y_\ell|^2 + |\nabla(\rho_1\partial_t - \mathcal{A}^{(1)})z_\ell|^2\right)\\ &+ (s\varphi_{\delta,1})^4\left(|\partial_t y_\ell|^2 + |\partial_t z_\ell|^2 + \sum_{i,j=1}^n |\partial_i\partial_j y_\ell|^2 + \sum_{i,j=1}^n|\partial_i\partial_j z_\ell|^2\right)\end{aligned}$$

$$+ (s\varphi_{\delta,1})^6 \left(|\nabla y_\ell|^2 + |\nabla z_\ell|^2\right) + (s\varphi_{\delta,1})^8 \left(|y_\ell|^2 + |z_\ell|^2\right)\Bigg] e^{2s\psi_{\delta,1}}\, dxdt$$

$$\le C \int_{Q_\delta} (s\varphi_{\delta,1})^2 \left(|\nabla \triangle a|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha a|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha c|^2 \right) e^{2s\psi_{\delta,1}}\, dxdt + C(s)B^2$$

for $\ell = 1, 2$.

Let us expand the left-hand side of (5.36). We have

$$\rho_2^2 \partial_t u_\ell(x,t) - \rho_1^2 \partial_t^2 u_\ell(x,t) + 2\rho_1 \partial_t \mathcal{A}^{(1)} u_\ell(x,t) - (\mathcal{A}^{(1)})^2 u_\ell(x,t) = F_\ell(x,t), \quad (x,t) \in Q,$$

for $\ell = 1, 2$. Moreover we have

$$\rho_2^2 \nabla \partial_t u_\ell(x,t) - \rho_1^2 \nabla \partial_t^2 u_\ell(x,t) + 2\rho_1 \nabla \partial_t \mathcal{A}^{(1)} u_\ell(x,t) \\ - \nabla (\mathcal{A}^{(1)})^2 u_\ell(x,t) = \nabla F_\ell(x,t), \quad (x,t) \in Q,$$

for $\ell = 1, 2$.

In particular at $t = t_0$, we have

$$\rho_2^2 \partial_t u_\ell(x,t_0) - \rho_1^2 \partial_t^2 u_\ell(x,t_0) + 2\rho_1 \partial_t \mathcal{A}^{(1)} u_\ell(x,t_0) \\ - (\mathcal{A}^{(1)})^2 u_\ell(x,t_0) = F_\ell(x,t_0),\ x \in \Omega, \tag{5.42}$$

and

$$\rho_2^2 \nabla \partial_t u_\ell(x,t_0) - \rho_1^2 \nabla \partial_t^2 u_\ell(x,t_0) + 2\rho_1 \nabla \partial_t \mathcal{A}^{(1)} u_\ell(x,t_0) \\ - \nabla (\mathcal{A}^{(1)})^2 u_\ell(x,t_0) = \nabla F_\ell(x,t_0),\ x \in \Omega, \tag{5.43}$$

for $\ell = 1, 2$. Taking the weighted L^2 norm of (5.42) and (5.43) in Ω, we obtain

$$\int_\Omega \left((s\varphi_{\delta,1})^2 |F_\ell(x,t_0)|^2 + |\nabla F_\ell(x,t_0)|^2 \right) e^{2s\psi_{\delta,1}(x,t_0)}\, dx \\ \le C \sum_{k=1}^6 J_{k,\ell} + C \int_\Omega \sum_{|\alpha|\le 5} (s\varphi_{\delta,1}(x,t_0))^2 |\partial_x^\alpha u_\ell(x,t_0)|^2 e^{2s\psi_{\delta,1}(x,t_0)}\, dx, \tag{5.44}$$

where

$$J_{1,\ell} = \int_\Omega (s\varphi_{\delta,1}(x,t_0))^2 |\partial_t u_\ell(x,t_0)|^2 e^{2s\psi_{\delta,1}(x,t_0)}\, dx,$$
$$J_{2,\ell} = \int_\Omega (s\varphi_{\delta,1}(x,t_0))^2 |\partial_t^2 u_\ell(x,t_0)|^2 e^{2s\psi_{\delta,1}(x,t_0)}\, dx,$$
$$J_{3,\ell} = \int_\Omega (s\varphi_{\delta,1}(x,t_0))^2 |\partial_t \mathcal{A}^{(1)} u_\ell(x,t_0)|^2 e^{2s\psi_{\delta,1}(x,t_0)}\, dx,$$
$$J_{4,\ell} = \int_\Omega |\nabla \partial_t u_\ell(x,t_0)|^2 e^{2s\psi_{\delta,1}(x,t_0)}\, dx,$$
$$J_{5,\ell} = \int_\Omega |\nabla \partial_t^2 u_\ell(x,t_0)|^2 e^{2s\psi_{\delta,1}(x,t_0)}\, dx,$$
$$J_{6,\ell} = \int_\Omega |\nabla \partial_t \mathcal{A}^{(1)} u_\ell(x,t_0)|^2 e^{2s\psi_{\delta,1}(x,t_0)}\, dx$$

for $\ell = 1, 2$.

Henceforth we assume that $s > 1$ is large enough to satisfy $s\varphi_{\delta,1} > 1$ in Q. We may estimate $J_{1,\ell}$ through $J_{6,\ell}$ for $\ell = 1, 2$, via the same argument which we used in the proof of Theorems 2.1 and 3.5 by using the inequality (5.41) derived by the Carleman estimate.

$$\begin{aligned} J_{1,\ell} &= \int_{t_0-\delta}^{t_0}\int_\Omega \partial_t\left(s^2\varphi_{\delta,1}^2|y_\ell|^2 e^{2s\psi_{\delta,1}}\right)\, dxdt \\ &\le C\int_{t_0-\delta}^{t_0}\int_\Omega \left[s^2\varphi_{\delta,1}^3|y_\ell|^2 + s^2\varphi_{\delta,1}^2|\partial_t y_\ell||y_\ell| + s^3\varphi_{\delta,1}^4|y_\ell|^2\right] e^{2s\psi_{\delta,1}}\, dxdt \\ &\le C\int_{Q_\delta} s^3\varphi_{\delta,1}^4\left(|y_\ell|^2 + |z_\ell|^2\right) e^{2s\psi_{\delta,1}}\, dxdt \end{aligned}$$

for $\ell = 1, 2$. Combining this with (5.41), we may estimate the right-hand side of the above inequality and we obtain

(5.45)

$$J_{1,\ell} \le C\int_{Q_\delta} \frac{\varphi_{\delta,1}^2}{s^3}\left(|\nabla\triangle a|^2 + \sum_{|\alpha|\le 2}|\partial_x^\alpha a|^2 + \sum_{|\alpha|\le 2}|\partial_x^\alpha c|^2\right) e^{2s\psi_{\delta,1}}\, dxdt + C(s)B^2$$

for $\ell = 1, 2$. Similarly, we obtain

$$\begin{aligned} J_{2,\ell} &= \int_{t_0-\delta}^{t_0}\int_\Omega \partial_t\left(s^2\varphi_{\delta,1}^2|\partial_t y_\ell|^2 e^{2s\psi_{\delta,1}}\right)\, dxdt \\ &\le C\int_{t_0-\delta}^{t_0}\int_\Omega \left[s^2\varphi_{\delta,1}^3|\partial_t y_\ell|^2 + s^2\varphi_{\delta,1}^2|\partial_t^2 y_\ell||\partial_t y_\ell| + s^3\varphi_{\delta,1}^4|\partial_t y_\ell|^2\right] e^{2s\psi_{\delta,1}}\, dxdt \\ &\le C\int_{Q_\delta} s^3\varphi_{\delta,1}^4\left(|\partial_t y_\ell|^2 + |\partial_t z_\ell|^2\right) e^{2s\psi_{\delta,1}}\, dxdt \end{aligned}$$

for $\ell = 1, 2$. Putting this together with (5.41), we see that

(5.46)

$$J_{2,\ell} \le C\int_{Q_\delta} s\varphi_{\delta,1}^2\left(|\nabla\triangle a|^2 + \sum_{|\alpha|\le 2}|\partial_x^\alpha a|^2 + \sum_{|\alpha|\le 2}|\partial_x^\alpha c|^2\right) e^{2s\psi_{\delta,1}}\, dxdt + C(s)B^2$$

for $\ell = 1, 2$. Moreover we have

$$\begin{aligned} J_{3,\ell} &= \int_{t_0-\delta}^{t_0}\int_\Omega \partial_t\left(s^2\varphi_{\delta,1}^2|\mathcal{A}^{(1)}y_\ell|^2 e^{2s\psi_{\delta,1}}\right)\, dxdt \\ &\le C\int_{t_0-\delta}^{t_0}\int_\Omega \Big[s^2\varphi_{\delta,1}^3|\mathcal{A}^{(1)}y_\ell|^2 \\ &\qquad\qquad + s^2\varphi_{\delta,1}^2|\partial_t\mathcal{A}^{(1)}y_\ell||\mathcal{A}^{(1)}y_\ell| + s^3\varphi_{\delta,1}^4|\mathcal{A}^{(1)}y_\ell|^2\Big] e^{2s\psi_{\delta,1}}\, dxdt \\ &\le C\int_{Q_\delta} s^3\varphi_{\delta,1}^4\left(|\mathcal{A}^{(1)}y_\ell|^2 + |\mathcal{A}^{(1)}z_\ell|^2\right) e^{2s\psi_{\delta,1}}\, dxdt \\ &\le C\int_{Q_\delta} s^3\varphi_{\delta,1}^4\left(\sum_{|\alpha|\le 2}|\partial_x^\alpha y_\ell|^2 + \sum_{|\alpha|\le 2}|\partial_x^\alpha z_\ell|^2\right) e^{2s\psi_{\delta,1}}\, dxdt \end{aligned}$$

for $\ell = 1, 2$. This together with (5.41) gives
(5.47)

$$J_{3,\ell} \le C \int_{Q_\delta} s\varphi_{\delta,1}^2 \left(|\nabla \triangle a|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha a|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha c|^2 \right) e^{2s\psi_{\delta,1}}\, dxdt + C(s)B^2$$

for $\ell = 1, 2$.

In the same way as getting the estimates of $\widetilde{J}_4, \widetilde{J}_5, \widetilde{J}_6$ in the proof of Theorem 3.5, we obtain

(5.48)

$$J_{4,\ell} \le \frac{C}{s^3} \int_{Q_\delta} \varphi_{\delta,1}^2 \left(|\nabla \triangle a|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha a|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha c|^2 \right) e^{2s\psi_{\delta,1}}\, dxdt + C(s)B^2,$$

(5.49)

$$J_{5,\ell} \le C \int_{Q_\delta} s\varphi_{\delta,1}^2 \left(|\nabla \triangle a|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha a|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha c|^2 \right) e^{2s\psi_{\delta,1}}\, dxdt + C(s)B^2,$$

(5.50)

$$J_{6,\ell} \le C \int_{Q_\delta} s\varphi_{\delta,1}^2 \left(|\nabla \triangle a|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha a|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha c|^2 \right) e^{2s\psi_{\delta,1}}\, dxdt + C(s)B^2$$

for $\ell = 1, 2$.

Summing up the estimate of (5.44) through (5.50), we have

$$\sum_{k=1}^{6} J_{k,\ell} \le C \int_{Q_\delta} s\varphi_{\delta,1}^2 \left(|\nabla \triangle a|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha a|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha c|^2 \right) e^{2s\psi_{\delta,1}}\, dxdt + C(s)B^2$$

for $\ell = 1, 2$.

This together with (5.44) gives

$$\begin{aligned} &\int_\Omega \left((s\varphi_{\delta,1})^2 |F_\ell(x,t_0)|^2 + |\nabla F_\ell(x,t_0)|^2 \right) e^{2s\psi_{\delta,1}(x,t_0)}\, dx \\ &\le C \int_{Q_\delta} s\varphi_{\delta,1}^2 \left(|\nabla \triangle a|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha a|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha c|^2 \right) e^{2s\psi_{\delta,1}}\, dxdt \\ &\quad + C \int_\Omega \sum_{|\alpha|\le 5} (s\varphi_{\delta,1}(x,t_0))^2 |\partial_x^\alpha u_\ell(x,t_0)|^2 e^{2s\psi_{\delta,1}(x,t_0)}\, dx + C(s)B^2 \end{aligned} \tag{5.51}$$

for $\ell = 1, 2$.

Let us estimate the left-hand side of the inequality (5.51) from below. By (5.37) at $t = t_0$, we have

$$\begin{aligned} &a^{(1)}(x)\nabla r_\ell(x,t_0) \cdot \nabla \triangle a(x) - r_\ell(x,t_0)\, \mathrm{div}(a^{(1)}(x))\nabla c(x)) \\ &\quad = F_\ell(x,t_0) - L_{1,\ell} a(x) - L_{2,\ell} c(x), \quad x \in \Omega, \end{aligned} \tag{5.52}$$

where

$$
\begin{aligned}
L_{1,\ell}a(x) &= 2a^{(1)}(x)\sum_{i,j=1}^{n}(\partial_i\partial_j r_\ell(x,t_0))(\partial_i\partial_j a(x)) \\
&\quad + a^{(1)}(x)\triangle r_\ell(x,t_0)\triangle a(x) + \nabla a^{(1)}(x)\cdot(\nabla r_\ell(x,t_0)\cdot\nabla)\nabla a(x) \\
&\quad + \Bigg[(\rho_2\partial_t^{\frac{1}{2}} - \rho_1\partial_t)\nabla r_\ell(x,t_0) + 3a^{(1)}(x)\nabla\triangle r_\ell(x,t_0) + (\triangle r_\ell(x,t_0))\,\nabla a^{(1)}(x) \\
&\qquad - c^{(1)}(x)\nabla r_\ell(x,t_0) + \frac{\rho_2\nabla r_\ell(x,0)}{\sqrt{\pi t_0}}\Bigg]\cdot\nabla a(x) \\
&\quad + \nabla a^{(1)}(x)\cdot(\nabla a(x)\cdot\nabla)\nabla r_\ell(x,t_0) \\
&\quad + \Bigg[(\rho_2\partial_t^{\frac{1}{2}} - \rho_1\partial_t)\triangle r_\ell(x,t_0) + \Big(\nabla a^{(1)}(x)\cdot\nabla\triangle r_\ell(x,t_0)\Big) \\
&\qquad + a^{(1)}(x)\triangle^2 r_\ell(x,t_0) - c^{(1)}(x)\triangle r_\ell(x,t_0) + \frac{\rho_2\triangle r_\ell(x,0)}{\sqrt{\pi t_0}}\Bigg]a(x), \quad x\in\Omega, \\
L_{2,\ell}c(x) &= -2a^{(1)}(x)\nabla r_\ell(x,t_0)\cdot\nabla c(x) \\
&\quad - \Bigg[(\rho_2\partial_t^{\frac{1}{2}} - \rho_1\partial_t)r_\ell(x,t_0) + \mathrm{div}(a^{(1)}(x)\nabla r_\ell(x,t_0)) \\
&\qquad - c^{(1)}(x)r_\ell(x,t_0) + \frac{\rho_2 r_\ell(x,0)}{\sqrt{\pi t_0}}\Bigg]c(x), \quad x\in\Omega,
\end{aligned}
$$

for $\ell = 1,2$. Multiplying (5.52) for $\ell = 1$ by $r_2(x,t_0)$, $x\in\Omega$, and multiplying (5.52) for $\ell = 2$ by $r_1(x,t_0)$, $x\in\Omega$, and then by the subtraction of new equations, we obtain

$$
\begin{aligned}
&a^{(1)}(x)\Big(r_2(x,t_0)\nabla r_1(x,t_0) - r_1(x,t_0)\nabla r_2(x,t_0)\Big)\cdot\nabla\triangle a(x) \\
&\quad = r_2(x,t_0)F_1(x,t_0) - r_1(x,t_0)F_2(x,t_0) \\
&\qquad - r_2(x,t_0)L_{1,1}a(x) + r_1(x,t_0)L_{1,2}a(x) \\
&\qquad - r_2(x,t_0)L_{2,1}c(x), + r_1(x,t_0)L_{2,2}c(x), \quad x\in\Omega.
\end{aligned} \tag{5.53}
$$

Note that

$$
\Big|\Big(r_2(x,t_0)\nabla r_1(x,t_0) - r_1(x,t_0)\nabla r_2(x,t_0)\Big)\cdot\nabla d_1(x)\Big| \geq m_3 > 0, \quad x\in\overline{\Omega},
$$

and $a\in H^4(\Omega)$ satisfies $a(x) = 0$, $x\in D$. Let us apply Lemma 4.8 to (5.53) in Ω. Then we obtain

$$
\begin{aligned}
&\int_\Omega\Bigg[s\varphi_{\delta,1}(x,t_0)\sum_{i,j,k=1}^{n}|\partial_i\partial_j\partial_k a(x)|^2 + (s\varphi_{\delta,1}(x,t_0))^2|\nabla\triangle a(x)|^2 \\
&\quad + (s\varphi_{\delta,1}(x,t_0))^3\sum_{i,j=1}^{n}|\partial_i\partial_j a(x)|^2 \\
&\quad + (s\varphi_{\delta,1}(x,t_0))^5\left(|\nabla a(x)|^2 + |a(x)|^2\right)\Bigg]e^{2s\psi_{\delta,1}(x,t_0)}\,dx
\end{aligned} \tag{5.54}
$$

$$\le C\sum_{\ell=1}^{2}\int_\Omega \left(|F_\ell(x,t_0)|^2 + |\nabla F_\ell(x,t_0)|^2\right) e^{2s\psi_{\delta,1}(x,t_0)}\,dx$$
$$+ C\int_\Omega \left(\sum_{|\alpha|\le 3}|\partial_x^\alpha a(x)|^2 + \sum_{|\alpha|\le 2}|\partial_x^\alpha c(x)|^2\right) e^{2s\psi_{\delta,1}(x,t_0)}\,dx.$$

On the other hand, by the equation (5.52) for $\ell = 1$ and $|r_1(x,t_0)| > 0$, $x \in \Omega$, we have

$$\text{(5.55)}\qquad \operatorname{div}(a^{(1)}(x))\nabla c(x)) - \frac{L_{2,1}c(x)}{r_1(x,t_0)}$$
$$= \frac{1}{r_1(x,t_0)}a^{(1)}(x)\nabla r_1(x,t_0)\cdot\nabla\triangle a(x) + \frac{L_{1,1}a(x)}{r_1(x,t_0)} - \frac{F_1(x,t_0)}{r_1(x,t_0)},\quad x\in\Omega.$$

Using the above equation (5.55) and the Carleman estimate for the elliptic equation stated in Lemma 4.6 ($p = 2$) for c, we get

$$\text{(5.56)}\qquad \int_\Omega s\varphi_{\delta,1}^2(x,t_0)\sum_{|\alpha|\le 2}|\partial_x^\alpha c|^2 e^{2s\psi_{\delta,1}(x,t_0)}\,dx$$
$$\le C\int_\Omega s\varphi_{\delta,1}(x,t_0)\sum_{|\alpha|\le 2}|\partial_x^\alpha c|^2 e^{2s\psi_{\delta,1}(x,t_0)}\,dx$$
$$\le C\int_\Omega \Bigg[s\varphi_{\delta,1}(x,t_0)\sum_{i,j=1}^{n}|\partial_i\partial_j c|^2$$
$$+ (s\varphi_{\delta,1}(x,t_0))^3|\nabla c|^2 + (s\varphi_{\delta,1}(x,t_0))^5|c|^2\Bigg] e^{2s\psi_{\delta,1}(x,t_0)}\,dx$$
$$\le C\int_\Omega (s\varphi_{\delta,1}(x,t_0))^2|F_1(x,t_0)|^2 e^{2s\psi_{\delta,1}(x,t_0)}\,dx$$
$$+ C\int_\Omega (s\varphi_{\delta,1}(x,t_0))^2\left(|\nabla\triangle a(x)|^2 + \sum_{|\alpha|\le 2}|\partial_x^\alpha a(x)|^2\right) e^{2s\psi_{\delta,1}(x,t_0)}\,dx.$$

By (5.54), we may estimate the second term on the right-hand side of (5.56) and we see that

$$\int_\Omega s\varphi_{\delta,1}^2(x,t_0)\sum_{|\alpha|\le 2}|\partial_x^\alpha c|^2 e^{2s\psi_{\delta,1}(x,t_0)}$$
$$\le C\sum_{\ell=1}^{2}\int_\Omega \left((s\varphi_{\delta,1}(x,t_0))^2|F_\ell(x,t_0)|^2 + |\nabla F_\ell(x,t_0)|^2\right) e^{2s\psi_{\delta,1}(x,t_0)}\,dx$$
$$+ C\int_\Omega \left(\sum_{|\alpha|\le 3}|\partial_x^\alpha a(x)|^2 + \sum_{|\alpha|\le 2}|\partial_x^\alpha c(x)|^2\right) e^{2s\psi_{\delta,1}(x,t_0)}\,dx.$$

This together with (5.54) gives

$$\int_\Omega \Bigg[s\varphi_{\delta,1}(x,t_0) \sum_{i,j,k=1}^n |\partial_i\partial_j\partial_k a(x)|^2 + (s\varphi_{\delta,1}(x,t_0))^2 |\nabla\triangle a(x)|^2 + (s\varphi_{\delta,1}(x,t_0))^3 \sum_{|\alpha|\le 2} |\partial_x^\alpha a(x)|^2 + s\varphi_{\delta,1}^2(x,t_0) \sum_{|\alpha|\le 2} |\partial_x^\alpha c|^2 \Bigg] e^{2s\psi_{\delta,1}(x,t_0)}\, dx$$
$$\le C \sum_{\ell=1}^2 \int_\Omega \left((s\varphi_{\delta,1}(x,t_0))^2 |F_\ell(x,t_0)|^2 + |\nabla F_\ell(x,t_0)|^2 \right) e^{2s\psi_{\delta,1}(x,t_0)}\, dx + C\int_\Omega \left(\sum_{|\alpha|\le 3} |\partial_x^\alpha a(x)|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha c(x)|^2 \right) e^{2s\psi_{\delta,1}(x,t_0)}\, dx.$$

Combining this with (5.51), we obtain

(5.57)
$$\int_\Omega \Bigg[s\varphi_{\delta,1}(x,t_0) \sum_{i,j,k=1}^n |\partial_i\partial_j\partial_k a(x)|^2 + (s\varphi_{\delta,1}(x,t_0))^2 |\nabla\triangle a(x)|^2 + (s\varphi_{\delta,1}(x,t_0))^3 \sum_{|\alpha|\le 2} |\partial_x^\alpha a(x)|^2 + s\varphi_{\delta,1}^2(x,t_0) \sum_{|\alpha|\le 2} |\partial_x^\alpha c|^2 \Bigg] e^{2s\psi_{\delta,1}(x,t_0)}\, dx$$
$$\le C\int_\Omega \left(\sum_{|\alpha|\le 3} |\partial_x^\alpha a(x)|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha c(x)|^2 \right) e^{2s\psi_{\delta,1}(x,t_0)}\, dx + C\int_{Q_\delta} s\varphi_{\delta,1}^2 \left(|\nabla\triangle a|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha a|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha c|^2 \right) e^{2s\psi_{\delta,1}}\, dxdt$$
$$+ C\sum_{\ell=1}^2 \int_\Omega \sum_{|\alpha|\le 5} (s\varphi_{\delta,1}(x,t_0))^2 |\partial_x^\alpha u_\ell(x,t_0)|^2 e^{2s\psi_{\delta,1}(x,t_0)}\, dx + C(s)B^2$$
$$\le C\int_\Omega \left[\sum_{i,j,k=1}^n |\partial_i\partial_j\partial_k a(x)|^2 + s\varphi_{\delta,1}^2(x,t_0) \Bigg(|\nabla\triangle a|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha a|^2 \Bigg) \right] e^{2s\psi_{\delta,1}(x,t_0)}\, dx$$
$$+ C\int_\Omega \sum_{|\alpha|\le 2} |\partial_x^\alpha c|^2 e^{2s\psi_{\delta,1}(x,t_0)}\, dx + C\int_{Q_\delta} s\varphi_{\delta,1}^2 \sum_{|\alpha|\le 2} |\partial_x^\alpha c|^2 e^{2s\psi_{\delta,1}}\, dxdt$$
$$+ C\sum_{\ell=1}^2 \int_\Omega \sum_{|\alpha|\le 5} (s\varphi_{\delta,1}(x,t_0))^2 |\partial_x^\alpha u_\ell(x,t_0)|^2 e^{2s\psi_{\delta,1}(x,t_0)}\, dx + C(s)B^2.$$

In the last inequality, we used

$$\varphi_{\delta,1}^2(x,t) e^{2s\psi_{\delta,1}(x,t)} \le \varphi_{\delta,1}^2(x,t_0) e^{2s\psi_{\delta,1}(x,t_0)}, \quad (x,t)\in Q_\delta,$$

for large $s>0$. As we have seen in the proof of Theorem 2.1, we may estimate the third term on the right-hand side of (5.57).

$$\int_{Q_\delta} s\varphi_{\delta,1}^2 \sum_{|\alpha|\le 2} |\partial_x^\alpha c|^2 e^{2s\psi_{\delta,1}}\, dxdt = \int_\Omega s\varphi_{\delta,1}^2(x,t_0) \sum_{|\alpha|\le 2} |\partial_x^\alpha c|^2 e^{2s\psi_{\delta,1}(x,t_0)} \widetilde{h}_s(x)\, dx,$$

where

$$\widetilde{h}_s(x) = \frac{1}{\varphi_{\delta,1}^2(x,t_0)} \int_{t_0-\delta}^{t_0+\delta} \varphi_{\delta,1}^2 e^{-2s(\psi_{\delta,1}(x,t_0)-\psi_{\delta,1}(x,t))}\, dt, \quad x \in \Omega,$$

and $\widetilde{h}_s$ converges uniformly to 0 in Ω.

Taking sufficiently large $s > 0$ and absorbing the first term through the third term on the right-hand side of (5.57) into the left-hand side, then we obtain

$$\begin{aligned}(5.58)\quad &\int_\Omega \Bigg[s\varphi_{\delta,1}(x,t_0) \sum_{i,j,k=1}^n |\partial_i\partial_j\partial_k a(x)|^2 + (s\varphi_{\delta,1}(x,t_0))^2 |\nabla\triangle a(x)|^2 \\ &\quad + (s\varphi_{\delta,1}(x,t_0))^3 \sum_{|\alpha|\le 2} |\partial_x^\alpha a(x)|^2 + s\varphi_{\delta,1}^2(x,t_0) \sum_{|\alpha|\le 2} |\partial_x^\alpha c|^2 \Bigg] e^{2s\psi_{\delta,1}(x,t_0)}\, dx \\ &\le C \sum_{\ell=1}^2 \int_\Omega \sum_{|\alpha|\le 5} (s\varphi_{\delta,1}(x,t_0))^2 |\partial_x^\alpha u_\ell(x,t_0)|^2 e^{2s\psi_{\delta,1}(x,t_0)}\, dx + C(s)B^2.\end{aligned}$$

Fix $s > 0$. Then we obtain

$$\|a\|_{H^3(\Omega)} + \|c\|_{H^2(\Omega)} \le C\|u(\cdot,t_0)\|_{H^5(\Omega)} + CB.$$

Thus we obtain the stability estimate (3.20). ∎

Proof of Theorem 3.9. We may obtain Theorem 3.9 in the same way as Theorem 3.8. ∎

Appendix A.

LEMMA A.1. *Let $u \in H^n(0,T)$. Then there exists a constant $C > 0$ such that*

$$\|\partial_t^\beta u\|_{L^2(0,T)} \le C\|\partial_t^n u\|_{L^2(0,T)}$$

for $n-1 < \beta < n$ ($n \in \mathbb{N}$).

Proof. Let us consider the Caputo derivative defined as

$$\partial_t^\beta u(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t \frac{\partial_\tau^n u(\tau)}{(t-\tau)^{\beta+1-n}}\, d\tau.$$

We note that $0 < n-\beta < 1$. We have

$$|\partial_t^\beta u|^2 \le \frac{1}{\Gamma(n-\beta)^2} \left[\int_0^t \frac{|\partial_\tau^n u|}{(t-\tau)^{\beta+1-n}}\, d\tau \right]^2.$$

Hence,

$$\begin{aligned}
\int_0^T |\partial_t^\beta u|^2\,dt &\le \frac{1}{\Gamma(n-\beta)^2}\int_0^T \left[\int_0^t \frac{1}{(t-\tau)^{\beta+1-n}}\,d\tau\right]\left[\int_0^t \frac{|\partial_\tau^n u|^2}{(t-\tau)^{\beta+1-n}}\,d\tau\right]dt \\
&= \frac{1}{\Gamma(n-\beta)^2}\int_0^T \frac{t^{n-\beta}}{n-\beta}\left[\int_0^t \frac{|\partial_\tau^n u|^2}{(t-\tau)^{\beta+1-n}}\,d\tau\right]dt \\
&\le \frac{1}{\Gamma(n-\beta)^2}\frac{T^{n-\beta}}{n-\beta}\int_0^T\int_0^t \frac{|\partial_\tau^n u|^2}{(t-\tau)^{\beta+1-n}}\,d\tau dt \\
&= \frac{1}{\Gamma(n-\beta)^2}\frac{T^{n-\beta}}{n-\beta}\int_0^T\int_\tau^T \frac{|\partial_\tau^n u|^2}{(t-\tau)^{\beta+1-n}}\,dt d\tau \\
&= \frac{1}{\Gamma(n-\beta)^2}\frac{T^{n-\beta}}{n-\beta}\int_0^T \frac{(T-\tau)^{n-\beta}}{n-\beta}|\partial_\tau^n u|^2\,d\tau \\
&\le \frac{1}{\Gamma(n-\beta)^2}\frac{T^{2(n-\beta)}}{(n-\beta)^2}\int_0^T |\partial_\tau^n u|^2\,d\tau.
\end{aligned}$$

Therefore there exists a constant $C > 0$ such that

$$\|\partial_t^\beta u\|_{L^2(0,T)} \le C\|\partial_t^n u\|_{L^2(0,T)}. \qquad \square$$

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