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The radiative transport equation in flatland with separation of variables

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The linear Boltzmann equation can be solved with separation of variables in one dimension, i.e., in three-dimensional space with planar symmetry. In this method, solutions are given by superpositions of eigenmodes which are sometimes called singular eigenfunctions. In this paper, we explore the singular-eigenfunction approach in flatland or two-dimensional space. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4958976]

I. INTRODUCTION

We consider the radiative transport equation or linear Boltzmann equation in flatland or in two spatial dimensions. The Green's function $G(\boldsymbol{\rho},\varphi;\varphi_0)$ ($\boldsymbol{\rho} = {}^t(x,y) \in \mathbb{R}^2$, $0 \le \varphi < 2\pi$, $0 \le \varphi_0 < 2\pi$) satisfies

$$\left(\hat{\boldsymbol{\Omega}} \cdot \nabla + 1\right) G(\boldsymbol{\rho},\varphi;\varphi_0) = \varpi \int_0^{2\pi} p(\varphi,\varphi') G(\boldsymbol{\rho},\varphi';\varphi_0)\, d\varphi' + \delta(\boldsymbol{\rho})\delta(\varphi-\varphi_0), \tag{1}$$

where $\hat{\boldsymbol{\Omega}} = {}^t(\cos\varphi, \sin\varphi)$ is a unit vector in $\mathbb{S}$, $\nabla = {}^t(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$, and $\varpi \in (0,1)$ is the albedo for single scattering. We have

$$G(\boldsymbol{\rho},\varphi;\varphi_0) \to 0 \quad \text{as} \quad |\boldsymbol{\rho}| \to \infty.$$

We suppose that the scattering phase function $p(\varphi,\varphi') \in L^\infty(\mathbb{S}\times\mathbb{S})$ is nonnegative and is normalized as

$$\int_0^{2\pi} p(\varphi,\varphi')\, d\varphi' = 1.$$

We assume that $p(\varphi,\varphi')$ is given by

$$\begin{aligned} p(\varphi,\varphi') &= \frac{1}{2\pi} \sum_{m=-L}^{L} \beta_m e^{im(\varphi-\varphi')} \\ &= \frac{1}{2\pi} + \frac{1}{\pi} \sum_{m=1}^{L} \beta_m \cos[m(\varphi-\varphi')], \end{aligned}$$

where $L \ge 0$, $\beta_0 = 1$, $-1 < \beta_l < 1$ ($l > 0$), and $\beta_{-m} = \beta_m$. We put

$$\beta_m = 0 \qquad \text{for} \quad |m| > L.$$

The Henyey-Greenstein model[21] is obtained by taking the limit $L \to \infty$ and putting $\beta_m = \mathrm{g}^{|m|}$ with a constant $\mathrm{g} \in (-1,1)$, where $\mathrm{g} = \int_0^{2\pi} \cos(\varphi-\varphi') p(\varphi,\varphi')\, d\varphi'$.

The radiative transport equation which depends on one spatial variable in three dimensions has attracted a lot of attention in linear transport theory. The singular-eigenfunction approach was explored as early as 1945 by Davison.[11] After further efforts such as Van Kampen,[48] Davison,[12] and

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Wigner,[50] Case established a way of finding solutions with separation of variables.[6,7] The method, called Case's method,[9,14] was soon extended to anisotropic scattering.[35,36]

On the other hand, the technique of rotated reference frames has existed in transport theory since 1964.[13,26] This method did not sound promising even though the idea was interesting. However, a decade ago, Markel succeeded in constructing an efficient numerical algorithm,[34] which is called the method of rotated reference frames,[31,38,41] to find solutions of the three-dimensional radiative transport equation by reinventing rotated reference frames.

Recently, the above two, separation of variables and rotated reference frames, were merged and Case's method was extended to three spatial variables.[32] There, rotated reference frames provide a tool to reduce three-dimensional equations to one dimension. With this tool, for example, the F_N method[43,44] was extended to three dimensions.[33]

The radiative transport equation is used in various subfields in science and engineering[1] such as light propagation in biological tissue,[2,4] clouds, and ocean,[45,47] seismic waves,[40] light in the interstellar medium,[10,39] neutron transport,[9] and remote sensing.[23] In these cases, usually three dimensions are most important. There are, however, cases where two dimensions have particular interests. Such flatland transport equations appear, for example, in wave scattering in the marginal ice zone[27] and wave transport along a surface with random impedance.[5] Sometimes optical tomography is considered in flatland.[3,19,20,25,46] The two-dimensional transport equation is also used for thermal radiative transfer[24] and heat transfer.[49] We note that the method of rotated reference frames was applied to two-dimensional space.[28–30]

In this paper, we consider the linear Boltzmann equation or radiative transport equation in flatland. Let μ denote the cosine of φ,

$$\mu = \cos\varphi, \qquad \varphi \in [0, 2\pi).$$

Let us introduce polynomials $\gamma_m(z)$ $(z \in \mathbb{C})$ which satisfy the following three-term recurrence relation:

$$2\nu h_m \gamma_m(\nu) - \gamma_{m+1}(\nu) - \gamma_{m-1}(\nu) = 0, \tag{2}$$

with initial terms

$$\gamma_0(\nu) = 1, \qquad \gamma_1(\nu) = (1 - \varpi)\nu.$$

Here,

$$h_m = 1 - \varpi\beta_m.$$

We have

$$\gamma_m(-\nu) = (-1)^m \gamma_m(\nu), \qquad \gamma_{-m}(\nu) = \gamma_m(\nu).$$

The function $g(z, \varphi)$ is given by

$$g(\nu, \varphi) = 1 + 2\sum_{m=1}^{L} \beta_m \gamma_m(\nu) \cos m\varphi. \tag{3}$$

We introduce

$$\Lambda(z) = 1 - \frac{\varpi z}{2\pi} \int_0^{2\pi} \frac{g(z, \varphi)}{z - \mu}\, d\varphi, \qquad z \in \mathbb{C} \setminus [-1, 1]. \tag{4}$$

Suppose $\Lambda(z)$ has $M = M(L, \varpi, \beta_m)$ positive roots. Let ν_j $(j = 0, \ldots, M-1)$ be positive roots which satisfy $\Lambda(\nu_j) = 0$. We further introduce

$$\lambda(\nu) = 1 - \frac{\varpi \nu}{2\pi} \mathcal{P} \int_0^{2\pi} \frac{g(\nu, \varphi)}{\nu - \mu}\, d\varphi, \qquad \nu \in (-1, 1), \tag{5}$$

where $\mathcal{P}$ denotes Cauchy's principal value. In Sec. II, we will see that singular eigenfunctions in flatland are obtained as

$$\phi(\nu, \varphi) = \frac{\varpi \nu}{2\pi} \mathcal{P} \frac{g(\nu, \varphi)}{\nu - \mu} + \frac{\sqrt{1 - \nu^2}}{2} \lambda(\nu) \delta(\nu - \mu), \tag{6}$$

where $\nu = \pm\nu_j$ $(j = 0, \dots, M-1)$ or $\nu \in (-1,1)$. Let us introduce the normalization factor

$$\mathcal{N}(\nu) = \begin{cases} \dfrac{2\nu}{\sqrt{1-\nu^2}}\left[\left(\dfrac{\varpi\nu}{2}\right)^2 g(\nu,\varphi_\nu)^2 + \lambda(\nu)^2\right], & \nu \in (-1,1), \\ \left(\dfrac{\varpi\nu}{2}\right)^2 g(\nu,\varphi_\nu)\dfrac{d\Lambda(\nu)}{d\nu}, & \nu \notin [-1,1]. \end{cases} \tag{7}$$

Here,

$$\varphi_\nu = \begin{cases} \cos^{-1}(\nu), & \nu \in [-1,1], \\ i\cosh^{-1}(\nu), & \nu > 1, \\ \pi + i\cosh^{-1}(|\nu|), & \nu < -1. \end{cases} \tag{8}$$

We note that $0 \le \cos^{-1}\nu \le \pi$ for $\nu \in [-1,1]$ and $\cosh^{-1}(|\nu|) = \ln\left(|\nu| + \sqrt{\nu^2-1}\right)$ for $|\nu| > 1$. Similarly, we use $\varphi_{\hat{\mathbf{k}}(\nu q)}$ for the analytically continued angle such that

$$\cos\varphi_{\hat{\mathbf{k}}(\nu q)} = \sqrt{1+(\nu q)^2}, \qquad \sin\varphi_{\hat{\mathbf{k}}(\nu q)} = -i\nu q,$$

for $\nu, q \in \mathbb{R}$. As is shown in Section V, the Green's function in flatland is obtained as

$$\begin{aligned} G(\boldsymbol{\rho},\varphi;\varphi_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iqy}\Bigg[\sum_{j=0}^{M-1}\phi(\pm\nu_j, \varphi - \varphi_{\hat{\mathbf{k}}(\pm\nu_j q)})\phi(\pm\nu_j, \varphi_0 - \varphi_{\hat{\mathbf{k}}(\pm\nu_j q)}) \\ \times \frac{1}{\sqrt{1+(\nu_j q)^2}\mathcal{N}(\nu_j)} e^{-\sqrt{1+(\nu_j q)^2}|x|/\nu_j} \\ + \int_0^1 \phi(\pm\nu, \varphi - \varphi_{\hat{\mathbf{k}}(\pm\nu q)})\phi(\pm\nu, \varphi_0 - \varphi_{\hat{\mathbf{k}}(\pm\nu q)}) \\ \times \frac{1}{\sqrt{1+(\nu q)^2}\mathcal{N}(\nu)} e^{-\sqrt{1+(\nu q)^2}|x|/\nu}\, d\nu\Bigg] dq, \end{aligned} \tag{9}$$

where upper signs are used for $x > 0$ and lower signs are used for $x < 0$.

The main purpose of the present paper is to derive (9). We will first consider the one-dimensional problem in two spatial dimensions with separation of variables in Sections II and III. Two-dimensional singular eigenfunctions are considered in Section IV. In particular their orthogonality relations are established. Then in Section V, we obtain the Green's function for the radiative transport equation in two dimensions by extending the one-dimensional problem to two dimensions using rotated reference frames. Finally, Section VII is devoted to concluding remarks. In Appendix, the Fourier-transform method is explained as an alternative approach.

II. ONE-DIMENSIONAL TRANSPORT THEORY IN FLATLAND

We begin with the one-dimensional homogeneous problem given by

$$\left(\mu\frac{\partial}{\partial x} + 1\right)\psi(x,\varphi) = \varpi\int_0^{2\pi} p(\varphi,\varphi')\psi(x,\varphi')\, d\varphi'. \tag{10}$$

We assume that solutions are given by the following form of separation of variables with separation constant ν:

$$\psi_\nu(x,\varphi) = \phi(\nu,\varphi)e^{-x/\nu}.$$

We normalize $\phi(\nu,\varphi)$ as

$$\int_0^{2\pi}\phi(\nu,\varphi)\, d\varphi = 1.$$

We then have

$$\left(1-\frac{\mu}{\nu}\right)\phi(\nu,\varphi)=\frac{\varpi}{2\pi}g(\nu,\varphi), \tag{11}$$

where

$$g(\nu,\varphi)=1+2\sum_{m=1}^{L}\beta_m\left[\gamma_m(\nu)\cos m\varphi+s_m(\nu)\sin m\varphi\right].$$

As is shown below, we have

$$\gamma_m(\nu)=\int_0^{2\pi}\phi(\nu,\varphi)\cos m\varphi\,d\varphi, \tag{12}$$

$$s_m(\nu)=\int_0^{2\pi}\phi(\nu,\varphi)\sin m\varphi\,d\varphi. \tag{13}$$

From (11), we obtain (6). Direct calculation shows that $\gamma_m(\nu)$ satisfy (2). Since (10) implies $\phi(\nu,-\varphi)=\phi(\nu,\varphi)$, coefficients for $\sin m\varphi$ should be zero. Indeed, $s_m(\nu)=0$ for all m as is shown below. For a function $f(\varphi)\in\mathbb{C}$, we have

$$\begin{aligned}\int_0^{2\pi}f(\varphi)\,d\varphi&=\int_0^{\pi}\left[f(\varphi)+f(\varphi+\pi)\right]\,d\varphi\\&=\int_{-1}^{1}\frac{f(\cos^{-1}\mu)+f(2\pi-\cos^{-1}\mu)}{\sqrt{1-\mu^2}}\,d\mu\\&=\int_{-1}^{1}\frac{f(\cos^{-1}\mu)+f(\pi+\cos^{-1}\mu)}{\sqrt{1-\mu^2}}\,d\mu,\end{aligned}$$

where we used $\int_0^{\pi}f(\varphi+\pi)\,d\varphi=\int_0^{\pi}f(2\pi-\varphi)\,d\varphi$. By using $\cos^{-1}(-\nu)=\pi-\cos^{-1}(\nu)$, we note that

$$\int_0^{2\pi}\delta(\nu-\mu)\sin(m\varphi)\,d\varphi=0,\qquad \nu\in\mathbb{R}.$$

If we plug (6) into (13), we obtain

$$s_m(\nu)=v_m(\nu)+\sum_{n=1}^{L}B_{mn}(\nu)s_n(\nu),$$

where

$$\begin{aligned}v_m(\nu)&=\frac{\varpi\nu}{2\pi}\mathcal{P}\int_0^{2\pi}\frac{\sin m\varphi}{\nu-\mu}\,d\varphi+\frac{\varpi\nu}{\pi}\sum_{n=1}^{L}\beta_n\mathcal{P}\int_0^{2\pi}\frac{\cos(n\varphi)\sin(m\varphi)}{\nu-\mu}\,d\varphi,\\B_{mn}(\nu)&=\frac{\varpi\nu}{\pi}\beta_n\mathcal{P}\int_0^{2\pi}\frac{\sin(n\varphi)\sin(m\varphi)}{\nu-\mu}\,d\varphi.\end{aligned}$$

However, we see that

$$\mathcal{P}\int_0^{2\pi}\frac{\cos(n\varphi)\sin(m\varphi)}{\nu-\mu}\,d\varphi=\mathcal{P}\int_0^{2\pi}\frac{\sin[(m+n)\varphi]+\sin[(m-n)\varphi]}{\nu-\mu}\,d\varphi=0,$$

for all m,n. Hence $v_m(\nu)=0$. Since the $L\times L$ matrix whose mn-element is given by $\delta_{mn}-B_{mn}(\nu)$ is invertible, we see that $s_m(v)=0$. Therefore we obtain (3) and singular eigenfunctions (6). We have

$$g(\nu,\varphi+2\pi)=g(\nu,\varphi)=g(\nu,-\varphi),\quad g(-\nu,\varphi)=g(\nu,\varphi+\pi).$$

When $\nu\in(-1,1)$, we obtain (5) by integrating (6) over φ. Note that from (5)

$$\lambda(-\nu)=1-\frac{\varpi\nu}{2\pi}\mathcal{P}\int_0^{2\pi}\frac{g(-\nu,\varphi)}{\nu+\mu}\,d\varphi=1-\frac{\varpi\nu}{2\pi}\mathcal{P}\int_0^{2\pi}\frac{g(\nu,\varphi)}{\nu-\mu}\,d\varphi=\lambda(\nu).$$

For $\nu \notin [-1,1]$ we have

$$1 = \int_0^{2\pi} \phi(\nu,\varphi)\, d\varphi = \frac{\varpi\nu}{2\pi} \int_0^{2\pi} \frac{g(\nu,\varphi)}{\nu-\mu}\, d\varphi.$$

Therefore discrete eigenvalues $\nu \in \mathbb{C} \setminus [-1,1]$ are roots of the function $\Lambda(\nu)$ given in (4). We note that if ν is a discrete eigenvalue, so is $-\nu$ because $\Lambda(-\nu) = \Lambda(\nu)$. That is, the eigenvalues $\pm\nu$ appear in pairs.

Singular eigenfunctions satisfy the following relations:

$$\phi(\nu,\varphi+2\pi) = \phi(\nu,\varphi) = \phi(\nu,-\varphi), \quad \phi(-\nu,\varphi) = \phi(\nu,\varphi+\pi).$$

Proposition 2.1 Discrete eigenvalues are real.

Proof. Let m_B be a positive integer such that $m_B \geq L$. We first show that if ν satisfies $\gamma_{m_B+1}(\nu) = 0$, then this ν is an eigenvalue of matrix B, which is the real symmetric matrix defined below. Hence $\nu \in \mathbb{R}$. Next we show that zeros of $\gamma_{m_B+1}(\nu)$ become roots of $\Lambda(\nu)$ as $m_B \to \infty$. With these two, the proof is completed.

Let us note that

$$|\beta_m| = \left|\beta_m e^{im\varphi}\right| = \left|\int_0^{2\pi} p(\varphi,\varphi') e^{im\varphi'}\, d\varphi'\right| \leq \int_0^{2\pi} p(\varphi,\varphi')\, d\varphi' = 1.$$

Hence $h_m > 0$ for all m. We can rewrite the three-term recurrence relation (2) as

$$b_m\left(\sqrt{2h_{m-1}}\gamma_{m-1}\right) + b_{m+1}\left(\sqrt{2h_{m+1}}\gamma_{m+1}\right) = \nu\left(\sqrt{2h_m}\gamma_m\right),$$

where

$$b_m = \frac{1}{2\sqrt{h_{m-1}h_m}}.$$

Similar to Ref. 18, we consider a tridiagonal $(2m_B+1)\times(2m_B+1)$ matrix $\bar{B}$ whose elements are given by

$$\begin{aligned}\bar{B}_{mm'} &= b_m\delta_{m',m-1} + b_{m+1}\delta_{m',m+1} \\ &\quad + b_{-m_B}\sqrt{\frac{h_{-m_B-1}}{h_{-m_B}}}\frac{\gamma_{-m_B-1}}{\gamma_{-m_B}}\delta_{m,-m_B}\delta_{m',-m_B} \\ &\quad + b_{m_B+1}\sqrt{\frac{h_{m_B+1}}{h_{m_B}}}\frac{\gamma_{m_B+1}}{\gamma_{m_B}}\delta_{m,m_B}\delta_{m',m_B},\end{aligned}$$

for $-m_B \leq m \leq m_B$ and $-m_B \leq m' \leq m_B$. Therefore if ν is a zero of $\gamma_{m_B+1}(\nu)$, we see that ν is an eigenvalue of the matrix B whose elements are given by

$$B_{mm'} = b_m\delta_{m',m-1} + b_{m+1}\delta_{m',m+1},$$

for $-m_B \leq m, m' \leq m_B$. Since B is real symmetric, ν is real. In particular we can say that $\nu \in \mathbb{R}$ even in the limit $m_B \to \infty$.[8,42]

Next we will explore the connection between roots of Λ and γ_{m_B+1} by repeatedly using the Christoffel-Darboux formula.[17,22]

In addition to (2), we introduce

$$2zp_m(z) - p_{m+1}(z) - p_{m-1}(z) = 0, \tag{14}$$

with

$$p_0(z) = 1, \qquad p_{\pm1} = z.$$

Furthermore we define

$$P_m(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos m\varphi}{z-\mu}\, d\varphi. \tag{15}$$

We have

$$2zP_m(z) - P_{m+1}(z) - P_{m-1}(z) = 2\delta_{m0}, \tag{16}$$

with

$$P_1(z) = zP_0(z) - 1.$$

Direct calculation of $\Lambda(z)$ in (4) shows

$$\Lambda(z) = 1 - \varpi z \left[P_0(z) + 2 \sum_{m=1}^{L} \beta_m \gamma_m(z) P_m(z) \right].$$

Let us consider $P_m(z) \times (2) - \gamma_m(z) \times (16)$. We have

$$\begin{aligned} -2\varpi z \beta_m P_m(z)\gamma_m(z) = &\left(P_m(z)\gamma_{m+1}(z) - P_{m+1}(z)\gamma_m(z) \right) \\ &- \left(P_{m-1}(z)\gamma_m(z) - P_m(z)\gamma_{m-1}(z) \right) \\ &- 2\delta_{m0}. \end{aligned}$$

We then take the summation on both sides by $\sum_{m=1}^{m_B}$. As a result we obtain

$$\Lambda(z) = P_{m_B}(z)\gamma_{m_B+1}(z) - P_{m_B+1}(z)\gamma_{m_B}(z). \tag{17}$$

Next, $\sum_{m=1}^{m_B} [p_m(z) \times (16) - P_m(z) \times (14)]$ yields

$$P_{m_B}(z)p_{m_B+1}(z) - P_{m_B+1}(z)p_{m_B}(z) = 1. \tag{18}$$

Moreover we obtain by $\sum_{m=1}^{m_B} [\gamma_m(z) \times (14) - p_m(z) \times (2)]$,

$$\varpi z g(z, \varphi_z) = \gamma_{m_B}(z)p_{m_B+1}(z) - \gamma_{m_B+1}(z)p_{m_B}(z). \tag{19}$$

By using (17)–(19), we obtain

$$\begin{aligned} p_{m_B+1}(z)\Lambda(z) &= p_{m_B+1}(z)P_{m_B}(z)\gamma_{m_B+1}(z) - p_{m_B+1}(z)P_{m_B+1}(z)\gamma_{m_B}(z) \\ &= \gamma_{m_B}(z) + \left[p_{m_B}(z)\gamma_{m_B+1}(z) - p_{m_B+1}(z)\gamma_{m_B}(z) \right] P_{m_B+1}(z) \\ &= \gamma_{m_B+1}(z) - \varpi z g(z, \varphi_z) P_{m_B+1}(z). \end{aligned}$$

Therefore we obtain

$$\Lambda(z) = \frac{\gamma_{m_B+1}(z)}{p_{m_B+1}(z)} - \varpi z g(z, \varphi_z) \frac{P_{m_B+1}(z)}{p_{m_B+1}(z)}.$$

However, $P_{m_B+1}(z)$ vanishes as $m_B \to \infty$ due to the Riemann-Lebesgue lemma. Thus discrete eigenvalues are zeros of γ_{m_B+1} as $m_B \to \infty$. □

We suppose there are $2M$ discrete eigenvalues $\pm\nu_j$ $(j = 0, \ldots, M-1)$ such that $\nu_j > 1$ and $\Lambda(\pm\nu_j) = 0$.

Definition 2.2. Let σ denote the set of "eigenvalues,"

$$\sigma = \{ \nu \in \mathbb{R};\ \nu \in (-1,1) \text{ or } \nu = \pm\nu_j,\ j = 0, 1, \ldots, M-1 \}.$$

For later calculations, we will prepare some notations. Let $\varphi_z \in \mathbb{C}$ be the angle such that

$$\Re\varphi_z \in [0, \pi], \qquad \cos\varphi_z = z, \qquad z \in \mathbb{C}.$$

When $\Im z = 0$ and we can write $z = \nu \in \mathbb{R}$, we have (8). In particular for $\nu \in (-1, 1)$, we obtain

$$g(-\nu, \varphi_{-\nu}) = g(-\nu, \pi - \varphi_\nu) = g(\nu, 2\pi - \varphi_\nu) = g(\nu, \varphi_\nu).$$

In the case that $\Re z = 0$, we have

$$\varphi_z = \frac{\pi}{2} - i \sinh^{-1}(\Im z) = \frac{\pi}{2} - i \ln \left(\Im z + \sqrt{(\Im z)^2 + 1} \right).$$

Suppose that $\Im z \neq 0$ and $\Re z \neq 0$. We obtain

$$\varphi_z = \tan^{-1}\left(\mathrm{sgn}(\Re z)\sqrt{\frac{1+|z|^2-r}{1-|z|^2+r}}\right) + i\ln\left(\frac{\sqrt{|z|^2+1+r}-\mathrm{sgn}(\Im z)\sqrt{|z|^2-1+r}}{\sqrt{2}}\right),$$

where

$$r = \sqrt{(|z|^2+1+2\Re z)(|z|^2+1-2\Re z)}.$$

Let ϵ be an infinitesimally small positive number. For $\nu \in (-1,1)$ we have

$$\begin{aligned}\Lambda^{\pm}(\nu) &:= \Lambda(\nu \pm i\epsilon)\\ &= \lambda(\nu) \pm i\frac{\varpi\nu}{2\sqrt{1-\nu^2}}\left[g(\nu,\varphi_\nu) + g(\nu, 2\pi-\varphi_\nu)\right]\\ &= \lambda(\nu) \pm i\frac{\varpi\nu}{\sqrt{1-\nu^2}}g(\nu,\varphi_\nu).\end{aligned}$$

We obtain

$$\Lambda^+(\nu) - \Lambda^-(\nu) = \frac{2i\varpi\nu}{\sqrt{1-\nu^2}}g(\nu,\varphi_\nu) = \frac{2i\varpi\nu}{\sqrt{1-\nu^2}}\left[1 + 2\sum_{m=1}^{L}\beta_m\gamma_m(\nu)\cos(m\varphi_\nu)\right],$$

and

$$\Lambda^+(\nu)\Lambda^-(\nu) = \lambda(\nu)^2 + \frac{\varpi^2\nu^2}{1-\nu^2}g(\nu,\varphi_\nu)^2.$$

We can estimate the number of discrete eigenvalues as follows:

Proposition 2.3. Suppose $\Lambda^+(\nu)\Lambda^-(\nu) \neq 0$ *for* $\nu \in [-1,1]$. *Then* $M \leq L+1$.

Proof. We prove the statement relying on the argument principle.[36] Since $\Lambda(z)$ is holomorphic in the whole plane cut between -1 and 1, according to the argument principle, the number of its roots is given by

$$2M = \frac{1}{2\pi}\Delta_C \arg\Lambda(z),$$

where Δ_C represents the change around the contour C which encircles the cut on the real axis from -1 to 1. Due to the assumed condition $\Lambda^+(\nu)\Lambda^-(\nu) \neq 0$, we have

$$2M = \frac{1}{2\pi}\left[\Delta_{C_+}\arg\Lambda^+(\nu) + \Delta_{C_-}\arg\Lambda^-(\nu)\right],$$

where $\Delta_{C_\pm}$ changes from 1 to -1 and from -1 to 1, respectively. Noting that

$$\Lambda^+(\nu) = \Lambda^-(-\nu), \qquad \arg\Lambda^+(\nu) = -\arg\Lambda^+(-\nu),$$

we finally obtain

$$M = \frac{1}{\pi}\Delta_{0\to1}\arg\Lambda^+(\nu),$$

where $\Delta_{0\to1}$ is the change when ν goes from 0 to 1. Note that

$$\arg\Lambda^+(0) = \arg\Lambda^-(0) = 0.$$

We see that M equals one plus the number that $\Lambda^+(\nu)$ crosses the real axis as ν moves from 0 to 1, or the number of roots of $\Im\Lambda^+(\nu)$. Since $g(\nu,\varphi_\nu)$ is an even polynomial of degree $2L$, there are at most L zeros on $(0,1)$. Therefore $M \leq L+1$. □

Remark 2.4. In the case of isotropic scattering ($L=0$), $\Lambda(z)$ is obtained as

$$\Lambda(z) = 1 - \frac{\varpi z}{\sqrt{z+1}\sqrt{z-1}}.$$

This $\Lambda(z)$ has only two roots $z = \pm\nu_0$ ($\nu_0 > 1$), i.e., $\Lambda(\pm\nu_0) = 0$, and we obtain

$$\nu_0 = \frac{1}{\sqrt{1-\varpi^2}}. \tag{20}$$

Thus the largest eigenvalue ν_0 can be explicitly written down in flatland. We note that in three dimensions with planar symmetry the largest eigenvalue is only obtained as a solution to the following transcendental equation:[7,9]

$$1 = \varpi\nu_0 \tanh^{-1}(1/\nu_0).$$

In the rest of this section, we explore orthogonality relations for $\phi(\nu,\varphi)$.

Lemma 2.5. Suppose $\nu_1, \nu_2 \in \sigma$ are different, i.e., $\nu_1 \neq \nu_2$. Then,

$$\int_0^{2\pi} \mu\phi(\nu_1,\varphi)\phi(\nu_2,\varphi)\,d\varphi = 0.$$

Proof. We consider the following two equations:

$$\left(1-\frac{\mu}{\nu_1}\right)\phi(\nu_1,\varphi) = \frac{\varpi}{2\pi} + \frac{\varpi}{\pi}\sum_{m=1}^{L}\beta_m\gamma_m(\nu_1)\cos m\varphi,$$

$$\left(1-\frac{\mu}{\nu_2}\right)\phi(\nu_2,\varphi) = \frac{\varpi}{2\pi} + \frac{\varpi}{\pi}\sum_{m=1}^{L}\beta_m\gamma_m(\nu_2)\cos m\varphi.$$

We multiply the upper equation by $\phi(\nu_2,\varphi)$ and the lower equation by $\phi(\nu_1,\varphi)$, integrate over φ, and subtract the second equation from the first equation. We obtain

$$\left(\frac{1}{\nu_2}-\frac{1}{\nu_1}\right)\int_0^{2\pi}\mu\phi(\nu_1,\varphi)\phi(\nu_2,\varphi)\,d\varphi = 0.$$

Thus the proof is completed. □

Theorem 2.6. *Consider $\nu,\nu' \in \sigma$. Let $\mathcal{N}(\nu)$ be the normalization factor in (7). We have*

$$\int_0^{2\pi}\mu\phi(\nu,\varphi)\phi(\nu',\varphi)\,d\varphi = \mathcal{N}(\nu)\delta(\nu-\nu').$$

Here the Dirac delta function $\delta(\nu-\nu')$ is read as the Kronecker delta $\delta_{\nu,\nu'}$ for $\nu,\nu' \notin [-1,1]$.

Proof. According to Lemma 2.5, the integral vanishes for $\nu \neq \nu'$. Hence it is enough if we show

$$\int_0^{2\pi}\mu\phi(\nu,\varphi)^2\,d\varphi = \mathcal{N}(\nu).$$

In the spirit of Ref. 35, we begin by defining

$$J(z,z') = \int_0^{2\pi}\mu\frac{g(z,\varphi)}{z-\mu}\frac{g(z',\varphi)}{z'-\mu}\,d\varphi,$$

for $z,z' \in \mathbb{C}\setminus[-1,1]$. We assume $z \neq z'$. We have

$$\begin{aligned} J(z,z') &= \frac{1}{z'-z}\int_0^{2\pi}\mu g(z,\varphi)g(z',\varphi)\left(\frac{1}{z-\mu}-\frac{1}{z'-\mu}\right)d\varphi \\ &= \frac{1}{z'-z}\left[\bar{\Gamma}_0(z)-\bar{\Gamma}_0(z')+2\sum_{m=1}^{L}\beta_m\Big(\gamma_m(z')\bar{\Gamma}_m(z)-\gamma_m(z)\bar{\Gamma}_m(z')\Big)\right], \end{aligned}$$

where

$$\bar{\Gamma}_m(z) = \int_0^{2\pi}\mu\frac{g(z,\varphi)}{z-\mu}\cos m\varphi\,d\varphi.$$

Let us write $\bar{\Gamma}_m(z)$ as

$$\bar{\Gamma}_m(z) = \Gamma_m(z) + \int_0^{2\pi} \mu \frac{g(\mu,\varphi)}{z-\mu} \cos m\varphi \, d\varphi, \tag{21}$$

where

$$\begin{aligned}\Gamma_m(z) &= \int_0^{2\pi} \mu \frac{g(z,\varphi) - g(\mu,\varphi)}{z-\mu} \cos m\varphi \, d\varphi \\ &= 2\sum_{n=1}^{L} \beta_n \int_0^{2\pi} \mu \frac{\gamma_n(z) - \gamma_n(\mu)}{z-\mu} \cos n\varphi \cos m\varphi \, d\varphi.\end{aligned}$$

The second term of the right-hand side of (21) is calculated as

$$\begin{aligned}&\int_0^{2\pi} \mu \frac{g(\mu,\varphi)}{z-\mu} \cos m\varphi \, d\varphi = \frac{1}{2i\varpi} \int_{-1}^{1} \frac{\Lambda^+(\mu) - \Lambda^-(\mu)}{z-\mu} \cos(m\varphi) \, d\mu \\ &= \frac{i}{2\varpi} \oint \frac{\Lambda(w)\cos(m\varphi_w)}{z-w} \, dw = \frac{\pi}{\varpi} \Lambda(z) \cos(m\varphi_z),\end{aligned}$$

where we chose the contour encircling the interval between -1 and 1. Hence we have

$$\bar{\Gamma}_m(z) = \Gamma_m(z) + \frac{\pi}{\varpi} \Lambda(z) \cos(m\varphi_z).$$

Therefore,

$$\begin{aligned}J(z,z') = \frac{1}{z'-z}\Big[\Gamma_0(z) - \Gamma_0(z') + 2\sum_{m=1}^{L} \beta_m \big(\gamma_m(z')\Gamma_m(z) - \gamma_m(z)\Gamma_m(z')\big)\Big] \\ + \frac{\pi}{\varpi} \frac{g(z',\varphi_z)\Lambda(z) - g(z,\varphi_{z'})\Lambda(z')}{z'-z}.\end{aligned}$$

On the right-hand side of the above equation, the first part vanishes. The last term can be rewritten as

$$\begin{aligned}\frac{g(z',\varphi_z)\Lambda(z) - g(z,\varphi_{z'})\Lambda(z')}{z'-z} &= \frac{g(z',\varphi_z) - g(z,\varphi_z)}{z'-z}\Lambda(z) - \frac{g(z,\varphi_{z'}) - g(z,\varphi_z)}{z'-z}\Lambda(z') \\ &\quad - g(z,\varphi_z)\frac{\Lambda(z') - \Lambda(z)}{z'-z}.\end{aligned}$$

Note that

$$\begin{aligned}\frac{g(z',\varphi_z) - g(z,\varphi_z)}{z'-z} &= 2\sum_{m=1}^{L} \beta_m \frac{\gamma_m(z') - \gamma_m(z)}{z'-z} \cos(m\varphi_z), \\ \frac{g(z,\varphi_{z'}) - g(z,\varphi_z)}{z'-z} &= 2\sum_{m=1}^{L} \beta_m \gamma_m(z) \frac{\cos(m\varphi_{z'}) - \cos(m\varphi_z)}{z'-z}.\end{aligned}$$

Let $\nu \notin [-1,1]$ be a discrete eigenvalue. We bring z to ν and then let z' approach ν. We obtain

$$\begin{aligned}\lim_{z'\to\nu}\lim_{z\to\nu} J(z,z') &= -g(\nu,\varphi_\nu) \lim_{z'\to\nu}\lim_{z\to\nu} \frac{\Lambda(z') - \Lambda(z)}{z'-z} \\ &= -g(\nu,\varphi_\nu)\frac{d\Lambda(\nu)}{d\nu}.\end{aligned}$$

That is,

$$\int_0^{2\pi} \mu\phi(\nu,\varphi)^2 \, d\varphi = \left(\frac{\varpi\nu}{2\pi}\right)^2 J(\nu,\nu) = \left(\frac{\varpi\nu}{2\pi}\right)^2 g(\nu,\varphi_\nu)\frac{d\Lambda(\nu)}{d\nu}.$$

Next we suppose that $\nu, \nu' \in (-1,1)$. We need to be careful about changing the order of integrals. Using the Poincare-Bertrand formula,[37]

$$\frac{\mathcal{P}}{\nu-\mu}\frac{\mathcal{P}}{\nu'-\mu} = \frac{1}{\nu-\nu'}\left(\frac{\mathcal{P}}{\nu'-\mu} - \frac{\mathcal{P}}{\nu-\mu}\right) + \pi^2\delta(\nu-\mu)\delta(\nu'-\mu),$$

we have

$$\begin{aligned}\int_0^{2\pi} \mu\phi(\nu,\varphi)\phi(\nu',\varphi)\,d\varphi &= \left(\frac{\varpi}{2\pi}\right)^2 \nu\nu' \int_0^{2\pi} \mu\mathcal{P}\frac{g(\nu,\varphi)}{\nu-\mu}\mathcal{P}\frac{g(\nu',\varphi)}{\nu'-\mu}\,d\varphi \\ &+ \frac{\varpi\nu}{2\pi}\lambda(\nu')\mathcal{P}\int_0^{2\pi} \mu\frac{g(\nu,\varphi)}{\nu-\mu}\delta(\nu'-\mu)\,d\varphi \\ &+ \frac{\varpi\nu'}{2\pi}\lambda(\nu)\mathcal{P}\int_0^{2\pi} \mu\frac{g(\nu',\varphi)}{\nu'-\mu}\delta(\nu-\mu)\,d\varphi \\ &+ \lambda(\nu)\lambda(\nu')\int_0^{2\pi} \mu\delta(\nu-\mu)\delta(\nu'-\mu)\,d\varphi.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_0^{2\pi} \mu\phi(\nu,\varphi)^2\,d\varphi \\ &= \lim_{\nu'\to\nu}\frac{1}{\nu-\nu'}\mathcal{P}\int_0^{2\pi}\mu\left[\frac{\varpi\nu}{2\pi}g(\nu,\varphi)\phi(\nu',\varphi) - \frac{\varpi\nu'}{2\pi}g(\nu',\varphi)\phi(\nu,\varphi)\right]d\varphi \\ &+ \delta(\nu-\nu')\lim_{\nu'\to\nu}\int_0^{2\pi}\mu\left[\frac{\varpi^2\nu\nu'}{4}g(\nu,\varphi)g(\nu',\varphi) + \lambda(\nu)\lambda(\nu')\right]\delta(\nu-\mu)\,d\varphi.\end{aligned}$$

The first term on the right-hand side vanishes. The second term on the right-hand side is computed as

$$\begin{aligned}&\lim_{\nu'\to\nu}\int_0^{2\pi}\mu\left[\frac{\varpi^2\nu\nu'}{4}g(\nu,\varphi)g(\nu',\varphi) + \lambda(\nu)\lambda(\nu')\right]\delta(\nu-\mu)\,d\varphi \\ &= \left[\left(\frac{\varpi\nu}{2}\right)^2 g(\nu,\varphi_\nu)^2 + \lambda(\nu)^2\right]\frac{2\nu}{\sqrt{1-\nu^2}}.\end{aligned}$$

□

III. ONE-DIMENSIONAL GREEN'S FUNCTION

Let us consider the Green's function $G(x,\varphi;\varphi_0)$ which satisfies

$$\left(\mu\frac{\partial}{\partial x}+1\right)G(x,\varphi;\varphi_0) = \varpi\int_0^{2\pi} p(\varphi,\varphi')G(x,\varphi';\varphi_0)\,d\varphi' + \delta(x)\delta(\varphi-\varphi_0),$$

and $G(x,\varphi;\varphi_0)\to 0$ as $|x|\to\infty$. The completeness of singular eigenfunctions can be shown in the usual way.[36] The Green's function is given by

$$\begin{cases} G(x,\varphi;\varphi_0) = \displaystyle\sum_{j=0}^{M-1} A_{j+}\psi_{\nu_j}(x,\varphi) + \int_0^1 A(\nu)\psi_\nu(x,\varphi)\,d\nu, & x>0, \\ G(x,\varphi;\varphi_0) = -\displaystyle\sum_{j=0}^{M-1} A_{j-}\psi_{-\nu_j}(x,\varphi) - \int_{-1}^0 A(\nu)\psi_\nu(x,\varphi)\,d\nu, & x<0, \end{cases}$$

with some coefficients $A_{j\pm}$ $(j=0,\dots,M-1)$ and $A(\nu)$. The jump condition is written as

$$G(0^+,\varphi;\varphi_0) - G(0^-,\varphi;\varphi_0) = \frac{1}{\mu}\delta(\varphi-\varphi_0).$$

Hence we have

$$\sum_{j=0}^{M-1}\left[A_{j+}\phi(\nu_j,\varphi) + A_{j-}\phi(-\nu_j,\varphi)\right] + \int_{-1}^1 A(\nu)\phi(\nu,\varphi)\,d\nu = \frac{1}{\mu}\delta(\varphi-\varphi_0).$$

Using orthogonality relations given in Theorem 2.6, the coefficients $A_{j\pm}$, $A(\nu)$ are determined as

$$A_{j\pm} = \frac{\phi(\pm\nu_j,\varphi_0)}{\mathcal{N}(\pm\nu_j)}, \qquad A(\nu) = \frac{\phi(\nu,\varphi_0)}{\mathcal{N}(\nu)}.$$

Therefore we obtain the one-dimensional Green's function as

$$G(x,\varphi;\varphi_0)=\sum_{j=0}^{M-1}\frac{\phi(\pm\nu_j,\varphi_0)\phi(\pm\nu_j,\varphi)}{\mathcal{N}(\nu_j)}e^{-|x|/\nu_j}+\int_0^1\frac{\phi(\pm\nu,\varphi_0)\phi(\pm\nu,\varphi)}{\mathcal{N}(\nu)}e^{-|x|/\nu}\,d\nu,$$

where upper signs are used for $x>0$ and lower signs are used for $x<0$.

IV. TWO-DIMENSIONAL TRANSPORT THEORY IN FLATLAND

To find the Green's function in (1), we consider the following homogeneous equation,

$$\left(\hat{\boldsymbol{\Omega}}\cdot\nabla+1\right)\psi(\boldsymbol{\rho},\varphi)=\varpi\int_0^{2\pi}p(\varphi,\varphi')\psi(\boldsymbol{\rho},\varphi')\,d\varphi'. \tag{22}$$

We consider rotation of the reference frame for some unit vector $\hat{\mathbf{k}}\in\mathbb{C}^2$ ($\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}=1$). By an operator $\mathcal{R}_{\hat{\mathbf{k}}}$, we measure angles in the reference frame whose x-axis lies in the direction of $\hat{\mathbf{k}}$. We have

$$\mathcal{R}_{\hat{\mathbf{k}}}\,\varphi=\varphi-\varphi_{\hat{\mathbf{k}}},$$

where $\varphi_{\hat{\mathbf{k}}}$ is the angle of $\hat{\mathbf{k}}$ in the laboratory frame. The dot product $\hat{\boldsymbol{\Omega}}\cdot\hat{\mathbf{k}}$ is expressed as

$$\hat{\boldsymbol{\Omega}}\cdot\hat{\mathbf{k}}=\mathcal{R}_{\hat{\mathbf{k}}}\,\mu.$$

We find that the inverse is given by

$$\mathcal{R}_{\hat{\mathbf{k}}}^{-1}\,\varphi=\varphi+\varphi_{\hat{\mathbf{k}}}.$$

Let us assume the angular flux $\psi(\boldsymbol{\rho},\varphi)$ has the form

$$\psi_\nu(\boldsymbol{\rho},\varphi)=\mathcal{R}_{\hat{\mathbf{k}}}\,\phi(\nu,\varphi)e^{-\hat{\mathbf{k}}\cdot\boldsymbol{\rho}/\nu}, \tag{23}$$

where ν is the separation constant. We will see that this $\phi(\nu,\varphi)$ is the singular eigenfunction developed in Section II.

By plugging (23) into (22), we obtain

$$\left(1-\frac{\hat{\boldsymbol{\Omega}}\cdot\hat{\mathbf{k}}}{\nu}\right)\mathcal{R}_{\hat{\mathbf{k}}}\,\phi(\nu,\varphi)=\varpi\int_0^{2\pi}\left[\frac{1}{2\pi}+\frac{1}{\pi}\sum_{m=1}^{L}\beta_m\cos\left(m(\mathcal{R}_{\hat{\mathbf{k}}}\,\varphi-\mathcal{R}_{\hat{\mathbf{k}}}\,\varphi')\right)\right]\mathcal{R}_{\hat{\mathbf{k}}}\,\phi(\nu,\varphi')\,d\varphi', \tag{24}$$

where we used $\varphi-\varphi'=\mathcal{R}_{\hat{\mathbf{k}}}\,\varphi-\mathcal{R}_{\hat{\mathbf{k}}}\,\varphi'$. By inverse rotation $\mathcal{R}_{\hat{\mathbf{k}}}^{-1}$, (24) reduces to (11). That is, $\mathcal{R}_{\hat{\mathbf{k}}}\,\phi(\nu,\varphi)$ is the singular eigenfunction for $\nu\in\sigma$ measured in the reference frame which is rotated by $\varphi_{\hat{\mathbf{k}}}$.

The unit vector $\hat{\mathbf{k}}$ is written as

$$\hat{\mathbf{k}}=\begin{pmatrix}\cos\varphi_{\hat{\mathbf{k}}}\\ \sin\varphi_{\hat{\mathbf{k}}}\end{pmatrix},$$

with angle $\varphi_{\hat{\mathbf{k}}}$. Let us set $\hat{\mathbf{k}}$ as

$$\hat{\mathbf{k}}=\hat{\mathbf{k}}(\nu q)=\begin{pmatrix}\hat{k}_x(\nu q)\\ -i\nu q\end{pmatrix},$$

where $q\in\mathbb{R}$ and

$$\hat{k}_x(\nu q)=\sqrt{1+(\nu q)^2}.$$

We will show orthogonality relations for $\mathcal{R}_{\hat{\mathbf{k}}}\,\phi(\nu,\varphi)$.

Theorem 4.1. *For $\nu,\nu'\in\sigma$ and any $q\in\mathbb{R}$ we have*

$$\int_0^{2\pi}\mu\left[\mathcal{R}_{\hat{\mathbf{k}}(\nu q)}\phi(\nu,\mu)\right]\left[\mathcal{R}_{\hat{\mathbf{k}}(\nu' q)}\,\phi(\nu',\mu)\right]\,d\varphi=\hat{k}_x(\nu q)\mathcal{N}(\nu)\delta(\nu-\nu'),$$

Proof. Similar to (24), we have

$$\left(1-\frac{\hat{\mathbf{\Omega}}\cdot\hat{\mathbf{k}}}{\nu}\right)\mathcal{R}_{\hat{\mathbf{k}}}\,\phi(\nu,\varphi)=\frac{\varpi}{2\pi}\int_0^{2\pi}\sum_{m=-L}^{L}\beta_m e^{im(\varphi-\varphi')}\mathcal{R}_{\hat{\mathbf{k}}}\,\phi(\nu,\varphi')\,d\varphi'.$$

For $\hat{\mathbf{k}}_1=\hat{\mathbf{k}}(\nu_1 q)$ and $\hat{\mathbf{k}}_2=\hat{\mathbf{k}}(\nu_2 q)$ with a fixed q, we have

$$\begin{aligned}\left[\mathcal{R}_{\hat{\mathbf{k}}_2}\phi(\nu_2,\varphi)\right]\mathcal{R}_{\hat{\mathbf{k}}_1}\left(1-\frac{\mu}{\nu_1}\right)\phi(\nu_1,\varphi)&=\frac{\varpi}{2\pi}\sum_{m=-L}^{L}\beta_m e^{im\varphi}\left[\mathcal{R}_{\hat{\mathbf{k}}_2}\phi(\nu_2,\varphi)\right]\\&\quad\times\int_0^{2\pi}e^{-im\varphi'}\mathcal{R}_{\hat{\mathbf{k}}_1}\phi(\nu_1,\varphi')\,d\varphi',\\\left[\mathcal{R}_{\hat{\mathbf{k}}_1}\phi(\nu_1,\varphi)\right]\mathcal{R}_{\hat{\mathbf{k}}_2}\left(1-\frac{\mu}{\nu_2}\right)\phi(\nu_2,\varphi)&=\frac{\varpi}{2\pi}\sum_{m=-L}^{L}\beta_m e^{-im\varphi}\left[\mathcal{R}_{\hat{\mathbf{k}}_1}\phi(\nu_1,\varphi)\right]\\&\quad\times\int_0^{2\pi}e^{im\varphi'}\mathcal{R}_{\hat{\mathbf{k}}_2}\phi(\nu_2,\varphi')\,d\varphi',\end{aligned}$$

where we used $\sum_m \beta_m e^{im(\varphi-\varphi')}=\sum_m \beta_m e^{-im(\varphi-\varphi')}$ ($\beta_{-m}=\beta_m$). By subtraction and integration from 0 to 2π, we obtain

$$\int_0^{2\pi}\left(\frac{\mathcal{R}_{\hat{\mathbf{k}}_2}\mu}{\nu_2}-\frac{\mathcal{R}_{\hat{\mathbf{k}}_1}\mu}{\nu_1}\right)\left[\mathcal{R}_{\hat{\mathbf{k}}_1}\phi(\nu_1,\varphi)\right]\left[\mathcal{R}_{\hat{\mathbf{k}}_2}\phi(\nu_2,\varphi)\right]\,d\varphi=0.$$

We note that

$$\mathcal{R}_{\hat{\mathbf{k}}}\,\mu=\hat{\mathbf{\Omega}}\cdot\hat{\mathbf{k}}=\hat{k}_x(\nu q)\cos\varphi-i\nu q\sin\varphi.$$

Thus,

$$\left(\frac{\hat{k}_x(\nu_2 q)}{\nu_2}-\frac{\hat{k}_x(\nu_1 q)}{\nu_1}\right)\int_0^{2\pi}\cos\varphi\left[\mathcal{R}_{\hat{\mathbf{k}}_1}\phi(\nu_1,\varphi)\right]\left[\mathcal{R}_{\hat{\mathbf{k}}_2}\phi(\nu_2,\varphi)\right]\,d\varphi=0.$$

We obtain

$$\int_0^{2\pi}\mu\left[\mathcal{R}_{\hat{\mathbf{k}}_1}\phi(\nu_1,\varphi)\right]\left[\mathcal{R}_{\hat{\mathbf{k}}_2}\phi(\nu_2,\varphi)\right]\,d\varphi=0,\qquad \nu_1\neq\nu_2.$$

When $\nu=\nu_1=\nu_2$, we can calculate the integral as

$$\begin{aligned}\int_0^{2\pi}\mu\left[\mathcal{R}_{\hat{\mathbf{k}}}\,\phi(\nu,\varphi)\right]^2 d\varphi&=\int_0^{2\pi}\left[\mathcal{R}_{\hat{\mathbf{k}}}^{-1}\mu\right]\phi(\nu,\varphi)^2\,d\varphi\\&=\hat{k}_x(\nu q)\int_0^{2\pi}\mu\phi(\nu,\varphi)^2\,d\varphi\\&=\hat{k}_x(\nu q)\mathcal{N}(\nu),\end{aligned}$$

where we used $\mathcal{R}_{\hat{\mathbf{k}}}^{-1}\,\mu=\cos(\mathcal{R}_{\hat{\mathbf{k}}}^{-1}\,\varphi)=\cos(\varphi+\varphi_{\hat{\mathbf{k}}})=\hat{k}_x(\nu q)\cos\varphi+i\nu q\sin\varphi$. Thus the orthogonality relations are proved. □

V. TWO-DIMENSIONAL GREEN'S FUNCTION

Let us consider the radiative transport equation (1). We can write the jump condition as

$$G(0^+,y,\varphi;\varphi_0)-G(0^-,y,\varphi;\varphi_0)=\frac{1}{\mu}\delta(y)\delta(\varphi-\varphi_0).$$

With the completeness of $\phi(\nu,\varphi)$ and plane-wave modes, the Green's function can be written as a superposition of $\psi_\nu(\boldsymbol{\rho},\varphi)$ in (23). Depending on x we can write

$$\begin{cases} G(\boldsymbol{\rho},\varphi;\varphi_0) = \displaystyle\int_{-\infty}^{\infty}\left[\sum_{j=0}^{M-1} A_{j+}(q)\psi_{\nu_j}(\boldsymbol{\rho},\varphi) + \int_0^1 A(\nu,q)\psi_\nu(\boldsymbol{\rho},\varphi)\,d\nu\right]\frac{dq}{2\pi}, & x>0,\\ G(\boldsymbol{\rho},\varphi;\varphi_0) = -\displaystyle\int_{-\infty}^{\infty}\sum_{j=0}^{M-1}\left[A_{j-}(q)\psi_{-\nu_j}(\boldsymbol{\rho},\varphi) + \int_{-1}^0 A(\nu,q)\psi_\nu(\boldsymbol{\rho},\varphi)\,d\nu\right]\frac{dq}{2\pi}, & x<0. \end{cases}$$

Here $A_{j\pm}(q)$, $A(\nu,q)$ are some coefficients. The jump condition reads

$$\begin{aligned} &\int_{-\infty}^{\infty} e^{iqy}\left[\sum_{j=0}^{M-1}\left(A_{j+}(q)\,\mathcal{R}_{\hat{\mathbf{k}}(\nu_j q)}\,\phi(\nu_j,\varphi) + A_{j-}(q)\,\mathcal{R}_{\hat{\mathbf{k}}(-\nu_j q)}\,\phi(-\nu_j,\varphi)\right)\right.\\ &\left.+\int_{-1}^1 A(\nu,q)\,\mathcal{R}_{\hat{\mathbf{k}}(\nu q)}\,\phi(\nu,\varphi)\,d\nu\right]\frac{dq}{2\pi}\\ &=\frac{1}{\mu}\delta(y)\delta(\varphi-\varphi_0). \end{aligned}$$

By using Theorem 4.1 of orthogonality relations, the coefficients $A_{j\pm}(q)$, $A(\nu,q)$ are determined as

$$A_{j\pm}(q) = \frac{\mathcal{R}_{\hat{\mathbf{k}}(\pm\nu_j q)}\,\phi(\pm\nu_j,\varphi_0)}{\hat{k}_x(\nu_j q)\mathcal{N}(\pm\nu_j)}, \quad A(\nu,q) = \frac{\mathcal{R}_{\hat{\mathbf{k}}(\nu q)}\,\phi(\nu,\varphi_0)}{\hat{k}_x(\nu q)\mathcal{N}(\nu)}.$$

Therefore the Green's function is obtained as

$$\begin{aligned} G(\boldsymbol{\rho},\varphi;\varphi_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iqy}\left[\sum_{j=0}^{M-1}\frac{\mathcal{R}_{\hat{\mathbf{k}}(\pm\nu_j q)}\,\phi(\pm\nu_j,\varphi_0)\phi(\pm\nu_j,\varphi)}{\hat{k}_x(\nu_j q)\mathcal{N}(\nu_j)}e^{-\hat{k}_x(\nu_j q)|x|/\nu_j}\right.\\ \left.+\int_0^1\frac{\mathcal{R}_{\hat{\mathbf{k}}(\pm\nu q)}\,\phi(\pm\nu,\varphi_0)\phi(\pm\nu,\varphi)}{\hat{k}_x(\nu q)\mathcal{N}(\nu)}e^{-\hat{k}_x(\nu q)|x|/\nu}\,d\nu\right]dq, \end{aligned}$$

where upper signs are used for $x>0$ and lower signs are used for $x<0$. The above Green's function can be rewritten as (9).

VI. ENERGY DENSITY

Let us calculate the energy density u for an isotropic source $\delta(\boldsymbol{\rho})$, i.e.,

$$u = \int_0^{2\pi}\int_0^{2\pi} G(\boldsymbol{\rho},\varphi;\varphi_0)\,d\varphi d\varphi_0.$$

Without loss of generality, we can put $y=0$ and assume $x>0$. Using (9), we obtain

$$u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\sum_{j=0}^{M-1}\frac{e^{-\hat{k}_x(\nu_j q)x/\nu_j}}{\hat{k}_x(\nu_j q)\mathcal{N}(\nu_j)} + \int_0^1\frac{e^{-\hat{k}_x(\nu q)x/\nu}}{\hat{k}_x(\nu q)\mathcal{N}(\nu)}\,d\nu\right]dq.$$

We note that

$$\int_{-\infty}^{\infty}\frac{e^{-\hat{k}_x(\nu q)x/\nu}}{\hat{k}_x(\nu q)}\,dq = \frac{2}{\nu}\int_0^{\infty}\frac{e^{-\sqrt{1+t^2}x/\nu}}{\sqrt{1+t^2}}\,dt = \frac{2}{\nu}\int_1^{\infty}\frac{e^{-sx/\nu}}{\sqrt{s^2-1}}\,ds = \frac{2}{\nu}K_0\left(\frac{x}{\nu}\right),$$

where K_0 is the modified Bessel function of the second kind of order zero. Hence we have

$$u(x) = \frac{1}{\pi}\left[\sum_{j=0}^{M-1}\frac{K_0(x/\nu_j)}{\nu_j\mathcal{N}(\nu_j)} + \int_0^1\frac{K_0(x/\nu)}{\nu\mathcal{N}(\nu)}\,d\nu\right]. \tag{25}$$

The expression (25) is the general result. Let us consider the case of isotropic scattering ($L = 0$). Since $g(\nu, \varphi) = 1$, we have

$$\Lambda(z) = 1 - \frac{\varpi z}{2\pi} \int_0^{2\pi} \frac{1}{z - \mu}\, d\varphi = 1 - \frac{\varpi z}{\sqrt{z^2 - 1}}, \quad z \in \mathbb{C} \setminus [-1, 1],$$

$$\lambda(\nu) = 1 - \frac{\varpi \nu}{2\pi} \mathcal{P} \int_0^{2\pi} \frac{1}{\nu - \mu}\, d\varphi = 1 - \frac{\varpi |\nu|}{\pi \sqrt{1 - \nu^2}} \left(\ln \left| \frac{\nu}{1 + \sqrt{1 - \nu^2}} \right| + \cosh^{-1} \frac{1}{\nu} \right),$$

where $\nu \in (-1, 1)$. Using the above $\Lambda(z)$ and $\lambda(\nu)$, we can calculate $\mathcal{N}(\nu)$ in (7). We note that $M = 1$ and the positive root ν_0 such that $\Lambda(\nu_0) = 0$ is given in (20).

VII. CONCLUDING REMARKS

We have obtained the Green's function for the radiative transport equation in flatland with separation of variables. As an alternative way, the Green's function can also be found with the Fourier transform. This calculation is summarized in the Appendix. Assuming the completeness of singular eigenfunctions in the presence of boundaries, the Green's function is given as a superposition of singular eigenfunctions, and the coefficients $A_{j\pm}(q), A(\nu, q)$ in Sec. V are determined from the boundary conditions. If we consider the present extension of Case's method, i.e., the separation of variables developed in this paper, in the planar geometry, we need the half-range completeness to express the Green's function as a superposition of singular eigenfunctions. Moreover we need to establish the half-range orthogonality relations to calculate $A_{j\pm}(q), A(\nu, q)$. It is another important future work to develop the separation of variables for the time-dependent radiative transport equation.

APPENDIX: FOURIER TRANSFORM

We will find an alternative expression of the Green's function (A5) by using the Fourier transform.[15,16]

Let us introduce the Fourier transform as

$$\tilde{G}(\mathbf{k}, \varphi; \varphi_0) = \int_{\mathbb{R}^2} e^{-i\mathbf{k}\cdot\boldsymbol{\rho}} G(\boldsymbol{\rho}, \varphi; \varphi_0)\, d\boldsymbol{\rho}.$$

By introducing

$$\tilde{G}_m(\mathbf{k}) = \int_0^{2\pi} \left[\mathcal{R}_{\hat{\mathbf{k}}}\, e^{-im\varphi} \right] \tilde{G}(\mathbf{k}, \varphi; \varphi_0)\, d\varphi, \tag{A1}$$

we can write (1) as

$$\left(1 + i\mathbf{k} \cdot \hat{\boldsymbol{\Omega}}\right) \tilde{G}(\mathbf{k}, \varphi; \varphi_0) = \frac{\varpi}{2\pi} \sum_{m=-L}^{L} \beta_m \left[\mathcal{R}_{\hat{\mathbf{k}}}\, e^{im\varphi} \right] \tilde{G}_m(\mathbf{k}) + \delta(\varphi - \varphi_0). \tag{A2}$$

Note that $P_m(z)$ in (15) can be written as

$$P_m(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{im\varphi}}{z - \mu}\, d\varphi.$$

Hereafter we set

$$z = \frac{i}{k}.$$

We then have

$$\tilde{G}_j(\mathbf{k}) = \varpi z \sum_{m=-L}^{L} \beta_m P_{m-j}(z) \tilde{G}_m(\mathbf{k}) + \frac{z}{z - \hat{\mathbf{k}} \cdot \hat{\boldsymbol{\Omega}}_0} \mathcal{R}_{\hat{\mathbf{k}}}\, e^{-ij\varphi_0}, \qquad |j| \le L. \tag{A3}$$

Hence

$$\sum_{m=-L}^{L}\left[\delta_{jm}-\varpi z\beta_m P_{m-j}(z)\right]\tilde{G}_m(\mathbf{k})=\frac{z}{z-\hat{\mathbf{k}}\cdot\hat{\mathbf{\Omega}}_0}\mathcal{R}_{\hat{\mathbf{k}}}\,e^{-ij\varphi_0},$$

for $|j|\le L$. Thus, by using (A2), $\tilde{G}(\mathbf{k},\varphi;\varphi_0)$ can be expressed using matrices as

$$\tilde{G}(\mathbf{k},\varphi;\varphi_0)=\frac{z}{z-\hat{\mathbf{k}}\cdot\hat{\mathbf{\Omega}}_0}\delta(\varphi-\varphi_0)+\frac{\varpi}{2\pi}\frac{z}{z-\hat{\mathbf{k}}\cdot\hat{\mathbf{\Omega}}}\frac{z}{z-\hat{\mathbf{k}}\cdot\hat{\mathbf{\Omega}}_0}\times\sum_{m=-L}^{L}\mathbf{P}^{\dagger}(\hat{\mathbf{k}},\varphi)\mathbf{W}[\mathbf{I}-\varpi z\mathbf{L}(z)\mathbf{W}]^{-1}\mathbf{P}(\hat{\mathbf{k}},\varphi_0).\tag{A4}$$

Here,

$$\{\mathbf{L}(z)\}_{jm}=P_{m-j}(z),\qquad\{\mathbf{W}\}_{jl}=\beta_m\delta_{jm},\qquad\{\mathbf{P}(\hat{\mathbf{k}},\varphi)\}_m=e^{-im(\varphi-\varphi_{\hat{\mathbf{k}}})}.$$

We note that

$$\int_{\mathbb{R}^2}\frac{e^{-\rho}}{\rho}\delta(\varphi_0-\varphi_{\hat{\boldsymbol{\rho}}})e^{-i\mathbf{k}\cdot\boldsymbol{\rho}}\,d\boldsymbol{\rho}=\frac{1}{1+i\mathbf{k}\cdot\hat{\mathbf{\Omega}}_0}.$$

Therefore we obtain the first alternative expression,

$$G(\boldsymbol{\rho},\varphi;\varphi_0)=\frac{e^{-\rho}}{\rho}\delta(\varphi_0-\varphi_{\hat{\boldsymbol{\rho}}})\delta(\varphi-\varphi_0)+\frac{\varpi}{(2\pi)^3}\int_{\mathbb{R}^2}e^{i\mathbf{k}\cdot\boldsymbol{\rho}}\frac{M(\mathbf{k},\varphi,\varphi_0)}{(1+i\mathbf{k}\cdot\hat{\mathbf{\Omega}})(1+i\mathbf{k}\cdot\hat{\mathbf{\Omega}}_0)}\,d\mathbf{k},\tag{A5}$$

where

$$M(\mathbf{k},\varphi,\varphi_0)=\sum_{m=-L}^{L}\mathbf{P}^{\dagger}(\hat{\mathbf{k}},\varphi)\mathbf{W}[\mathbf{I}-\varpi z\mathbf{L}(z)\mathbf{W}]^{-1}\mathbf{P}(\hat{\mathbf{k}},\varphi_0).$$

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