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The Green's function for the three-dimensional linear Boltzmann equation via Fourier transform

Manabu Machida

Department of Mathematical Sciences, The University of Tokyo, Komaba, Meguro, Tokyo 153-8914, Japan

E-mail: mmachida@ms.u-tokyo.ac.jp

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Abstract

The linear Boltzmann equation with constant coefficients in the three-dimensional infinite space is revisited. It is known that the Green's function can be calculated via the Fourier transform in the case of isotropic scattering. In this paper, we show that the three-dimensional Green's function can be computed with the Fourier transform even in the case of arbitrary anisotropic scattering.

Keywords: Boltzmann equation, linear transport theory, radiative transfer

1. Introduction

We consider the linear Boltzmann equation in three-dimensions, which governs neutron transport and radiative transfer. If scattering is isotropic, it is well known that the Green's function of the monoenergetic neutron transport in a three-dimensional infinite medium can be obtained using the Fourier transform [1, 11]. In one-dimension, Ganapol developed Fourier transform techniques and showed that the Green's function can be found even for arbitrary anisotropic scattering [5, 8]. In this paper, we will extend Ganapol's calculation in one-dimensional transport theory to three-dimensions making use of rotated reference frames and present the three-dimensional Green's function for arbitrary anisotropic scattering.

The introduction of rotated reference frames in neutron transport theory goes back to Dede [3] and Kobayashi [12]. Dede discussed that three-dimensional equations in the P_N method reduce to one-dimensional equations by measuring angles in the reference frame rotated in the direction of the Fourier vector. Kobayashi's work is similar to the calculation in the present paper in the sense that the recurrence relation (15) was derived, however, $\bar{\psi}_l^m$ was not explicitly obtained as we will do in (20). The first practical way of using rotated reference frames, which made numerical calculation possible, was found by Markel [15]. Markel established an efficient method of computing the specific intensity of light in three-dimensions by expressing the specific intensity in terms of eigenmodes and rotating the reference frame for each eigenmode [15, 17]. The technique was also applied to inverse transport problems [18]. Recently, it was found that the use of such rotated reference frames is not restricted to Legendre polynomials and spherical harmonics. Case's singular eigenfunctions were extended to three-dimensions [13]. With this result, the F_N method [19, 20] was extended to three-dimensions [14].

Let us write the Green's function of monoenergetic neutron transport in three-dimensions. The angular flux $G(\mathbf{r}, \hat{\mathbf{\Omega}}; \hat{\mathbf{\Omega}}_0) \in \mathbb{R}$ ($\mathbf{r} \in \mathbb{R}^3$, $\hat{\mathbf{\Omega}} \in \mathbb{S}^2$) obeys

$$(\hat{\mathbf{\Omega}} \cdot \nabla + 1)G(\mathbf{r}, \hat{\mathbf{\Omega}}; \hat{\mathbf{\Omega}}_0) = c\int_{\mathbb{S}^2} p(\hat{\mathbf{\Omega}}, \hat{\mathbf{\Omega}}')G(\mathbf{r}, \hat{\mathbf{\Omega}}'; \hat{\mathbf{\Omega}}_0)\,\mathrm{d}\hat{\mathbf{\Omega}}' + \delta(\mathbf{r})\delta(\hat{\mathbf{\Omega}} - \hat{\mathbf{\Omega}}_0),$$

where c $(0 < c < 1)$ is the albedo for single scattering and $p(\hat{\mathbf{\Omega}}, \hat{\mathbf{\Omega}}') \in \mathbb{R}$ is the phase function. The unit vector $\hat{\mathbf{\Omega}}$ has the polar angle θ and azimuthal angle φ. The source placed at the origin $\mathbf{r} = \mathbf{0}$ emits neutrons in the direction $\hat{\mathbf{\Omega}}_0$. We assume $p(\hat{\mathbf{\Omega}}, \hat{\mathbf{\Omega}}')$ depends only on $\hat{\mathbf{\Omega}} \cdot \hat{\mathbf{\Omega}}'$ and write

$$p(\hat{\mathbf{\Omega}}, \hat{\mathbf{\Omega}}') = \sum_{l=0}^{L}\sum_{m=-l}^{l}\frac{\beta_l}{2l+1}Y_{lm}(\hat{\mathbf{\Omega}})Y_{lm}^*(\hat{\mathbf{\Omega}}') = \frac{1}{4\pi}\sum_{l=0}^{L}\sum_{m=-l}^{l}\omega_l^m P_l^m(\mu)P_l^m(\mu')\mathrm{e}^{\mathrm{i}m(\varphi-\varphi')},$$

where $\beta_0 = 1$, $|\beta_l| < 2l + 1$ $(l = 1, 2, \ldots, L)$, and we defined $\mu = \cos\theta$ and

$$\omega_l^m = \beta_l\frac{(l-m)!}{(l+m)!}.$$

Here $Y_{lm}(\hat{\mathbf{\Omega}})$ are spherical harmonics given by

$$Y_{lm}(\hat{\mathbf{\Omega}}) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\mu)\mathrm{e}^{\mathrm{i}m\varphi}$$

and

$$P_l^m(\mu) = (-1)^m(1-\mu^2)^{m/2}\frac{\mathrm{d}^m}{\mathrm{d}\mu^m}P_l(\mu), \qquad P_l^{-m}(\mu) = (-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(\mu),$$

for $0 \leqslant m \leqslant l$. Associated Legendre polynomials $P_l^m(\mu)$ satisfy the following recurrence relation.

$$(2l+1)\mu P_l^m(\mu) = (l-m+1)P_{l+1}^m(\mu) + (l+m)P_{l-1}^m(\mu), \tag{1}$$

with initial terms

$$P_m^m(\mu) = (-1)^m(2m-1)!!(1-\mu^2)^{m/2}, \qquad P_{m+1}^m(\mu) = (2m+1)\mu P_m^m(\mu),$$

for $0 \leqslant m \leqslant l$.

The first two terms in the collision expansion of the Green's function contain the Dirac delta function. In particular the first term $G_0(\mathbf{r}, \hat{\mathbf{\Omega}}; \hat{\mathbf{\Omega}}_0)$ expresses uncollided particles which travel in the medium without experiencing scattering. We subtract the uncollided part as follows.

$$G(\mathbf{r}, \hat{\mathbf{\Omega}}; \hat{\mathbf{\Omega}}_0) = G_0(\mathbf{r}, \hat{\mathbf{\Omega}}; \hat{\mathbf{\Omega}}_0) + \psi(\mathbf{r}, \hat{\mathbf{\Omega}}),$$

where $G_0(\mathbf{r}, \hat{\mathbf{\Omega}}; \hat{\mathbf{\Omega}}_0)$ satisfies

$$(\hat{\mathbf{\Omega}} \cdot \nabla + 1)G_0(\mathbf{r}, \hat{\mathbf{\Omega}}; \hat{\mathbf{\Omega}}_0) = \delta(\hat{\mathbf{\Omega}} - \hat{\mathbf{\Omega}}_0)\delta(\mathbf{r})$$

and $\psi(\mathbf{r}, \hat{\mathbf{\Omega}})$ satisfies

$$(\hat{\mathbf{\Omega}} \cdot \nabla + 1)\psi(\mathbf{r}, \hat{\mathbf{\Omega}}) = c\int_{\mathbb{S}^2} p(\hat{\mathbf{\Omega}}, \hat{\mathbf{\Omega}}')\psi(\mathbf{r}, \hat{\mathbf{\Omega}}')\,\mathrm{d}\hat{\mathbf{\Omega}}' + \frac{c}{r^2}\mathrm{e}^{-r}p(\hat{\mathbf{\Omega}}, \hat{\mathbf{\Omega}}_0)\delta(\hat{\mathbf{r}} - \hat{\mathbf{\Omega}}_0),$$

where

$$r = |\mathbf{r}|, \qquad \hat{\mathbf{r}} = \frac{\mathbf{r}}{r}.$$

The source term in the transport equation for $\psi(\mathbf{r}, \hat{\mathbf{\Omega}})$ can be calculated by noting

$$\begin{aligned} c\int_{\mathbb{S}^2} p(\hat{\mathbf{\Omega}}, \hat{\mathbf{\Omega}}')G_0(\mathbf{r}, \hat{\mathbf{\Omega}}'; \hat{\mathbf{\Omega}}_0)\,\mathrm{d}\hat{\mathbf{\Omega}}' &= c\int_{\mathbb{S}^2} p(\hat{\mathbf{\Omega}}, \hat{\mathbf{\Omega}}')\frac{1}{r^2}\mathrm{e}^{-r}\delta(\hat{\mathbf{\Omega}}' - \hat{\mathbf{r}})\delta(\hat{\mathbf{\Omega}}' - \hat{\mathbf{\Omega}}_0)\,\mathrm{d}\hat{\mathbf{\Omega}}' \\ &= \frac{c}{r^2}\mathrm{e}^{-r}p(\hat{\mathbf{\Omega}}, \hat{\mathbf{\Omega}}_0)\delta(\hat{\mathbf{r}} - \hat{\mathbf{\Omega}}_0), \end{aligned}$$

where we used

$$G_0(\mathbf{r}, \hat{\mathbf{\Omega}}; \hat{\mathbf{\Omega}}_0) = \frac{1}{r^2}\mathrm{e}^{-r}\delta(\hat{\mathbf{\Omega}} - \hat{\mathbf{r}})\delta(\hat{\mathbf{\Omega}} - \hat{\mathbf{\Omega}}_0).$$

In this paper we will consider how $\psi(\mathbf{r}, \hat{\mathbf{\Omega}})$ is obtained.

In the case of isotropic scattering ($L = 0$), we can compute $\psi(\mathbf{r}, \hat{\mathbf{\Omega}})$ with the textbook way (appendix A) as

$$\psi(\mathbf{r}, \hat{\mathbf{\Omega}}) = \frac{c}{2(2\pi)^4}\int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\frac{\left[1 - \frac{c}{k}\tan^{-1}(k)\right]^{-1}}{(1 + \mathrm{i}\mathbf{k}\cdot\hat{\mathbf{\Omega}})(1 + \mathrm{i}\mathbf{k}\cdot\hat{\mathbf{\Omega}}_0)}\,\mathrm{d}\mathbf{k}.$$

The aim of this paper is to extend this result to arbitrary anisotropic scattering.

As the first main result, we obtain $\psi(\mathbf{r}, \hat{\mathbf{\Omega}})$ as

$$\psi(\mathbf{r}, \hat{\mathbf{\Omega}}) = \frac{c}{2(2\pi)^4}\int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\frac{M(\mathbf{k}, \hat{\mathbf{\Omega}}, \hat{\mathbf{\Omega}}_0)}{(1 + \mathrm{i}\mathbf{k}\cdot\hat{\mathbf{\Omega}})(1 + \mathrm{i}\mathbf{k}\cdot\hat{\mathbf{\Omega}}_0)}\,\mathrm{d}\mathbf{k}, \tag{2}$$

where $M(\mathbf{k}, \hat{\mathbf{\Omega}}, \hat{\mathbf{\Omega}}_0)$ is given in (11).

Since the calculation of $M(\mathbf{k}, \hat{\mathbf{\Omega}}, \hat{\mathbf{\Omega}}_0)$ involves matrix inversion, we explore an alternative expression of $\psi(\mathbf{r}, \hat{\mathbf{\Omega}})$. As the second main result, we will show that $\psi(\mathbf{r}, \hat{\mathbf{\Omega}})$ is given by

$$\psi(\mathbf{r}, \hat{\mathbf{\Omega}}) = \frac{1}{(2\pi)^3}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}(\hat{\mathbf{\Omega}})\int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\mathrm{e}^{-\mathrm{i}m\varphi_{\hat{\mathbf{k}}}}\kappa_{lm}(\mathbf{k})\,\mathrm{d}\mathbf{k}, \tag{3}$$

where

$$\begin{aligned} \kappa_{lm}(\mathbf{k}) = \sum_{m'=-l}^{l}\frac{1}{g_{|m'|}^{m'}(\mathrm{i}/k)}\sqrt{\frac{2l+1}{4\pi}\frac{(l-m')!}{(l+m')!}}\mathrm{e}^{-\mathrm{i}m'\varphi_0} \\ \times d_{mm'}^{\,l}(\theta_{\hat{\mathbf{k}}})\left[g_l^{m'}\left(\frac{\mathrm{i}}{k}\right)\bar{\psi}_{|m'|}^{m'}\left(\frac{\mathrm{i}}{k}, \hat{\mathbf{k}}\right) + \chi_l^{m'}\left(\frac{\mathrm{i}}{k}\right)\right]. \end{aligned}$$

Here, $d_{m'm}^{\,l}$ are Wigner's d-matrices [22] and g_l^m are Chandrasekhar's polynomials of the first kind (see section 4) [2, 4]. Below, χ_l^m and $\bar{\psi}_{|m|}^m$ are given in (21) and (22), respectively.

Suppose that $\bar{\psi}_{|m|}^m$ is independent of $\varphi_{\hat{\mathbf{k}}}$. We can write $\kappa_{lm}(\mathbf{k}) = \kappa_{lm}(k, \mu_{\hat{\mathbf{k}}})$. In this case we obtain

$$\psi(\mathbf{r},\,\hat{\mathbf{\Omega}}) = \frac{1}{(2\pi)^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}(\hat{\mathbf{\Omega}})\mathrm{i}^m\int_0^\infty k^2 \times \int_{-1}^{1} J_m\left(kr\sqrt{1-\mu_{\hat{\mathbf{k}}}^2}\sin\theta_{\hat{\mathbf{r}}}\right)\mathrm{e}^{\mathrm{i}kr\mu_{\hat{\mathbf{k}}}\cos\theta_{\hat{\mathbf{r}}}}\mathrm{e}^{-\mathrm{i}m\varphi_{\hat{\mathbf{r}}}}\kappa_{lm}(k,\,\mu_{\hat{\mathbf{k}}})\,\mathrm{d}\mu_{\hat{\mathbf{k}}}\mathrm{d}k, \tag{4}$$

where J_m is the Bessel function of degree m. For example, $\kappa_{lm}(k,\,\hat{\mathbf{k}})$ is independent of $\varphi_{\hat{\mathbf{k}}}$ if $\hat{\mathbf{\Omega}}_0 = \hat{\mathbf{z}}$.

In what follows, we derive (2) in section 3 and (3) in section 4. In section 5, we compute the energy density by using (4). The key idea of rotated reference frames is introduced in the next section.

2. Rotated reference frames

We introduce the operator $\mathcal{R}_{\hat{\mathbf{k}}} : \mathbb{C} \mapsto \mathbb{C}$ for a unit vector $\hat{\mathbf{k}} \in \mathbb{C}^3$ ($\hat{\mathbf{k}}\cdot\hat{\mathbf{k}} = 1$). By operating $\mathcal{R}_{\hat{\mathbf{k}}}$ we measure $\hat{\mathbf{\Omega}}$ in the reference frame whose z-axis lies in the direction of $\hat{\mathbf{k}}$ [15]. For example, we have

$$\hat{\mathbf{\Omega}}\cdot\hat{\mathbf{k}} = \mathcal{R}_{\hat{\mathbf{k}}}\mu.$$

If a function $f(\hat{\mathbf{\Omega}}) \in \mathbb{C}$ is given as

$$f(\hat{\mathbf{\Omega}}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_{lm}Y_{lm}(\hat{\mathbf{\Omega}}),$$

we have

$$\mathcal{R}_{\hat{\mathbf{k}}}f(\hat{\mathbf{\Omega}}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_{lm}\sum_{m'=-l}^{l}\mathrm{e}^{-\mathrm{i}m'\varphi_{\hat{\mathbf{k}}}}d^{\,l}_{m'm}(\theta_{\hat{\mathbf{k}}})Y_{lm'}(\hat{\mathbf{\Omega}}),$$

where $\theta_{\hat{\mathbf{k}}}$ and $\varphi_{\hat{\mathbf{k}}}$ are the polar and azimuthal angles of $\hat{\mathbf{k}}$ in the laboratory frame. In particular, we have

$$\mathcal{R}_{\hat{\mathbf{k}}}Y_{lm}(\hat{\mathbf{\Omega}}) = \sum_{m'=-l}^{l}\mathrm{e}^{-\mathrm{i}m'\varphi_{\hat{\mathbf{k}}}}d^{\,l}_{m'm}(\theta_{\hat{\mathbf{k}}})Y_{lm'}(\hat{\mathbf{\Omega}}).$$

3. Fourier transform

We begin by noting that $\hat{\mathbf{\Omega}}\cdot\hat{\mathbf{\Omega}}' = (\mathcal{R}_{\hat{\mathbf{k}}}\hat{\mathbf{\Omega}})\cdot(\mathcal{R}_{\hat{\mathbf{k}}}\hat{\mathbf{\Omega}}')$ and

$$p(\hat{\mathbf{\Omega}},\,\hat{\mathbf{\Omega}}') = \sum_{l=0}^{L}\sum_{m=-l}^{l}\frac{\beta_l}{2l+1}[\mathcal{R}_{\hat{\mathbf{k}}}Y_{lm}(\hat{\mathbf{\Omega}})][\mathcal{R}_{\hat{\mathbf{k}}}Y_{lm}^*(\hat{\mathbf{\Omega}}')],$$

for an arbitrary unit vector $\hat{\mathbf{k}}$. The transport equation is written as

$$(\hat{\mathbf{\Omega}}\cdot\nabla + 1)G(\mathbf{r},\,\hat{\mathbf{\Omega}};\,\hat{\mathbf{\Omega}}_0) = c\sum_{l=0}^{L}\sum_{m=-l}^{l}\frac{\beta_l}{2l+1}[\mathcal{R}_{\hat{\mathbf{k}}}Y_{lm}(\hat{\mathbf{\Omega}})] \times \int_{\mathbb{S}^2}[\mathcal{R}_{\hat{\mathbf{k}}}Y_{lm}^*(\hat{\mathbf{\Omega}}')]G(\mathbf{r},\,\hat{\mathbf{\Omega}}';\,\hat{\mathbf{\Omega}}_0)\,\mathrm{d}\hat{\mathbf{\Omega}}' + \delta(\mathbf{r})\delta(\hat{\mathbf{\Omega}} - \hat{\mathbf{\Omega}}_0).$$

By taking projections of the Green's function with rotated spherical harmonics, we introduce

$$G_l^m(\mathbf{r}) = \sqrt{\frac{4\pi}{2l+1}\frac{(l+m)!}{(l-m)!}}\,\mathrm{e}^{\mathrm{i}m\varphi_0}\int_{\mathbb{S}^2}[\mathcal{R}_{\hat{\mathbf{k}}}Y_{lm}^*(\hat{\mathbf{\Omega}})]G(\mathbf{r},\hat{\mathbf{\Omega}};\hat{\mathbf{\Omega}}_0)\,\mathrm{d}\hat{\mathbf{\Omega}}.$$

An arbitrary vector $\mathbf{k}\in\mathbb{R}^3$ is given by k $(0\leqslant k<\infty)$ and $\hat{\mathbf{k}}\in\mathbb{R}^3$ $(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}=1)$ as

$$\mathbf{k}=k\hat{\mathbf{k}}.$$

With this vector $\mathbf{k}$ we perform the Fourier transform as

$$\bar{G}(\mathbf{k},\hat{\mathbf{\Omega}};\hat{\mathbf{\Omega}}_0)=\int_{\mathbb{R}^3}\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}G(\mathbf{r},\hat{\mathbf{\Omega}};\hat{\mathbf{\Omega}}_0)\,\mathrm{d}\mathbf{r},$$
$$\bar{G}_l^m(\mathbf{k})=\int_{\mathbb{R}^3}\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}G_l^m(\mathbf{r})\,\mathrm{d}\mathbf{r}.$$

In the Fourier space we obtain

$$(1+\mathrm{i}\mathbf{k}\cdot\hat{\mathbf{\Omega}})\bar{G}(\mathbf{k},\hat{\mathbf{\Omega}})=c\sum_{l=0}^{L}\sum_{m=-l}^{l}\frac{\omega_l^m}{\sqrt{4\pi(2l+1)}}\sqrt{\frac{(l+m)!}{(l-m)!}}\,\mathrm{e}^{-\mathrm{i}m\varphi_0}[\mathcal{R}_{\hat{\mathbf{k}}}Y_{lm}(\hat{\mathbf{\Omega}})]\bar{G}_l^m(\mathbf{k})$$
$$+\,\delta(\hat{\mathbf{\Omega}}-\hat{\mathbf{\Omega}}_0).$$

This is expressed as

$$\bar{G}(\mathbf{k},\hat{\mathbf{\Omega}})=\frac{c}{2}\frac{1}{1+\mathrm{i}\mathbf{k}\cdot\hat{\mathbf{\Omega}}}\sum_{l=0}^{L}\sum_{m=-l}^{l}\frac{\omega_l^m}{\sqrt{\pi(2l+1)}}\sqrt{\frac{(l+m)!}{(l-m)!}}\,\mathrm{e}^{-\mathrm{i}m\varphi_0}[\mathcal{R}_{\hat{\mathbf{k}}}Y_{lm}(\hat{\mathbf{\Omega}})]\bar{G}_l^m(\mathbf{k})$$
$$+\frac{1}{1+\mathrm{i}\mathbf{k}\cdot\hat{\mathbf{\Omega}}}\delta(\hat{\mathbf{\Omega}}-\hat{\mathbf{\Omega}}_0). \tag{5}$$

By multiplying $\mathcal{R}_{\hat{\mathbf{k}}}Y_{lm}^*(\hat{\mathbf{\Omega}})$ on both sides of (5) and integrating over $\mathbb{S}^2$, we obtain

$$\int_{\mathbb{S}^2}[\mathcal{R}_{\hat{\mathbf{k}}}Y_{lm}^*(\hat{\mathbf{\Omega}})]\bar{G}(\mathbf{k},\hat{\mathbf{\Omega}})\,\mathrm{d}\hat{\mathbf{\Omega}}=\frac{c}{2}\sum_{l'=0}^{L}\sum_{m'=-l'}^{l'}\frac{\omega_{l'}^{m'}}{\sqrt{\pi(2l'+1)}}\sqrt{\frac{(l'+m')!}{(l'-m')!}}\,\mathrm{e}^{-\mathrm{i}m'\varphi_0}\bar{G}_{l'}^{m'}(\mathbf{k})$$
$$\times\int_{\mathbb{S}^2}\frac{1}{1+\mathrm{i}\mathbf{k}\cdot\hat{\mathbf{\Omega}}}[\mathcal{R}_{\hat{\mathbf{k}}}Y_{lm}^*(\hat{\mathbf{\Omega}})][\mathcal{R}_{\hat{\mathbf{k}}}Y_{l'm'}(\hat{\mathbf{\Omega}})]\,\mathrm{d}\hat{\mathbf{\Omega}}$$
$$+\frac{\mathcal{R}_{\hat{\mathbf{k}}}Y_{lm}^*(\hat{\mathbf{\Omega}}_0)}{1+\mathrm{i}\mathbf{k}\cdot\hat{\mathbf{\Omega}}_0}.$$

We note that

$$\mathbf{k}\cdot\hat{\mathbf{\Omega}}=k\mathcal{R}_{\hat{\mathbf{k}}}\mu.$$

We put

$$z=\frac{\mathrm{i}}{k}.$$

Let us define

$$L_{jl}^m(z)=\frac{z}{2}\int_{-1}^{1}\frac{P_j^m(\mu)P_l^m(\mu)}{z-\mu}\,\mathrm{d}\mu.$$

We note that

$$L_{jl}^m(z)=zQ_l^m(z)P_j^m(z),\qquad j\leqslant l, \tag{6}$$

where $P_l^m(z)$ and $Q_l^m(z)$ are associated Legendre functions of the first and second kinds which have a branch cut from $-\infty$ to 1. For $m\geqslant 0$, they satisfy the recurrence relation (1) with initial terms

$$P_m^m(z) = (2m-1)!!(z-1)^{m/2}(z+1)^{m/2}, \qquad P_{m+1}^m(z) = (2m+1)zP_m^m(z)$$

and

$$Q_m^m(z) = \frac{[(2m-1)!!]^2}{2P_m^m(z)}\int_{-1}^{1}\frac{(1-\mu^2)^m}{z-\mu}\,\mathrm{d}\mu, \qquad Q_{m+1}^m(z) = (2m+1)zQ_m^m(z) - \frac{(2m)!}{P_m^m(z)}.$$

Let $\Theta(\cdot)$ be the step function such that $\Theta(x) = 1$ for $x \geqslant 0$ and $\Theta(x) = 0$ otherwise. We have

$$\bar{G}_j^m(\mathbf{k}) = \Theta(L-|m|)c\sum_{l=|m|}^{L}\omega_l^m L_{jl}^m(z)\bar{G}_l^m(\mathbf{k}) + \mathrm{e}^{\mathrm{i}m\varphi_0}\mathcal{R}_{\hat{\mathbf{k}}}\frac{z}{z-\mu_0}P_j^m(\mu_0)\mathrm{e}^{-\mathrm{i}m\varphi_0}. \tag{7}$$

We note that the information on the phase function $p(\hat{\boldsymbol{\Omega}}, \hat{\boldsymbol{\Omega}}')$ is embedded in ω_l^m in the first term on the right-hand side of (7). For $|m| \leqslant j \leqslant L$, the above equation can be rewritten as

$$\sum_{l=|m|}^{L}[\delta_{jl} - c\omega_l^m L_{jl}^m(z)]\bar{G}_l^m(\mathbf{k}) = \frac{z}{z-\hat{\mathbf{k}}\cdot\hat{\boldsymbol{\Omega}}_0}P_j^m(\hat{\mathbf{k}}\cdot\hat{\boldsymbol{\Omega}}_0)\mathrm{e}^{\mathrm{i}m\varphi_0}\mathcal{R}_{\hat{\mathbf{k}}}\mathrm{e}^{-\mathrm{i}m\varphi_0}. \tag{8}$$

As we will see below, an exact solution is readily obtained from (5) and (8). We introduce the following matrices and vectors.

$$\begin{aligned}
\{\mathbf{L}^m(z)\}_{jl} &= L_{jl}^m(z),\\
\{\mathbf{W}^m\}_{jl} &= \omega_l^m\delta_{jl},\\
\{\bar{\mathbf{G}}^m(\mathbf{k})\}_l &= \bar{G}_l^m(\mathbf{k}),\\
\{\mathbf{P}^m(\hat{\mathbf{k}}, \hat{\boldsymbol{\Omega}})\}_j &= P_j^m(\hat{\mathbf{k}}\cdot\hat{\boldsymbol{\Omega}})\mathrm{e}^{\mathrm{i}m\varphi_0}\mathcal{R}_{\hat{\mathbf{k}}}\mathrm{e}^{-\mathrm{i}m\varphi}.
\end{aligned}$$

We then have

$$[\mathbf{I} - c\mathbf{L}^m(z)\mathbf{W}^m]\bar{\mathbf{G}}^m(\mathbf{k}) = \frac{z}{z-\hat{\mathbf{k}}\cdot\hat{\boldsymbol{\Omega}}_0}\mathbf{P}^m(\hat{\mathbf{k}}, \hat{\boldsymbol{\Omega}}_0),$$

where $\mathbf{I}$ is the identity. Using (5), $\bar{G}(\mathbf{k}, \hat{\boldsymbol{\Omega}})$ is given by

$$\bar{G}(\mathbf{k}, \hat{\boldsymbol{\Omega}}) = \frac{z}{z-\hat{\mathbf{k}}\cdot\hat{\boldsymbol{\Omega}}_0}\delta(\hat{\boldsymbol{\Omega}} - \hat{\boldsymbol{\Omega}}_0) + \frac{c}{4\pi}\frac{z}{z-\hat{\mathbf{k}}\cdot\hat{\boldsymbol{\Omega}}}\sum_{m=-L}^{L}\mathbf{P}^m(\hat{\mathbf{k}}, \hat{\boldsymbol{\Omega}})^\dagger\mathbf{W}^m\bar{\mathbf{G}}^m(\mathbf{k}).$$

Therefore we obtain

$$\begin{aligned}
G(\mathbf{r}, \hat{\boldsymbol{\Omega}}) = {}& \frac{\mathrm{e}^{-r}}{r^2}\delta(\hat{\boldsymbol{\Omega}} - \hat{\mathbf{r}})\delta(\hat{\boldsymbol{\Omega}} - \hat{\boldsymbol{\Omega}}_0)\\
&+ \frac{c}{2(2\pi)^4}\int_{\mathbb{R}^3}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\frac{z}{z-\hat{\mathbf{k}}\cdot\hat{\boldsymbol{\Omega}}}\frac{z}{z-\hat{\mathbf{k}}\cdot\hat{\boldsymbol{\Omega}}_0}\\
&\times\sum_{m=-L}^{L}\mathbf{P}^m(\hat{\mathbf{k}}, \hat{\boldsymbol{\Omega}})^\dagger\mathbf{W}^m[\mathbf{I} - c\mathbf{L}^m(z)\mathbf{W}^m]^{-1}\mathbf{P}^m(\hat{\mathbf{k}}, \hat{\boldsymbol{\Omega}}_0)\,\mathrm{d}\mathbf{k}.
\end{aligned} \tag{9}$$

Here we used

$$\int_{\mathbb{R}^3}\frac{\mathrm{e}^{-r}}{r^2}\delta\left(\hat{\boldsymbol{\Omega}} - \frac{\mathbf{r}}{r}\right)\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\,\mathrm{d}\mathbf{r} = \frac{1}{1+\mathrm{i}\mathbf{k}\cdot\hat{\boldsymbol{\Omega}}}. \tag{10}$$

We note that the first term in the above equation is the uncollided term $G_0(\mathbf{r}, \hat{\boldsymbol{\Omega}}; \hat{\boldsymbol{\Omega}}_0)$ in the collision expansion. The uncollided part is naturally singled out in the present formulation aiming at the first main result (2). By defining

$$M(\mathbf{k},\,\hat{\mathbf{\Omega}},\,\hat{\mathbf{\Omega}}_0)=\sum_{m=-L}^{L}\mathbf{P}^m(\hat{\mathbf{k}},\,\hat{\mathbf{\Omega}})^{\dagger}\mathbf{W}^m[\mathbf{I}-c\mathbf{L}^m(z)\mathbf{W}^m]^{-1}\mathbf{P}^m(\hat{\mathbf{k}},\,\hat{\mathbf{\Omega}}_0),\tag{11}$$

we obtain (2). If $L=0$, we have

$$M(\mathbf{k},\,\hat{\mathbf{\Omega}},\,\hat{\mathbf{\Omega}}_0)=\frac{1}{1-cL_{00}^0(z)}=\frac{1}{1-c\frac{\mathrm{i}}{k}Q_0\left(\frac{\mathrm{i}}{k}\right)}=\frac{1}{1-\frac{c}{k}\tan^{-1}(k)},$$

where we used (6) and (26).

4. Nonstandard Fourier transform

Let us explore an alternative formulation which is potentially more suitable for numerical calculation. Similarly to the previous section, we will closely follow [8]. By focusing on the collided part $\psi(\mathbf{r},\,\hat{\mathbf{\Omega}})$ we have

$$\begin{aligned}(\hat{\mathbf{\Omega}}\cdot\nabla+1)\psi(\mathbf{r},\,\hat{\mathbf{\Omega}})=&\,c\sum_{l=0}^{L}\sum_{m=-l}^{l}\frac{\beta_l}{2l+1}[\mathcal{R}_{\hat{\mathbf{k}}}Y_{lm}(\hat{\mathbf{\Omega}})]\int_{\mathbb{S}^2}[\mathcal{R}_{\hat{\mathbf{k}}}Y_{lm}^*(\hat{\mathbf{\Omega}}')]\psi(\mathbf{r},\,\hat{\mathbf{\Omega}}')\,\mathrm{d}\hat{\mathbf{\Omega}}'\\&+\frac{c}{r^2}\mathrm{e}^{-r}p(\hat{\mathbf{\Omega}},\,\hat{\mathbf{\Omega}}_0)\delta(\hat{\mathbf{r}}-\hat{\mathbf{\Omega}}_0).\end{aligned}$$

Let us take projections of $\psi(\mathbf{r},\,\hat{\mathbf{\Omega}})$ with rotated spherical harmonics as

$$\psi_l^m(\mathbf{r})=\sqrt{\frac{4\pi}{2l+1}\frac{(l+m)!}{(l-m)!}}\mathrm{e}^{\mathrm{i}m\varphi_0}\int_{\mathbb{S}^2}[\mathcal{R}_{\hat{\mathbf{k}}}Y_{lm}^*(\hat{\mathbf{\Omega}})]\psi(\mathbf{r},\,\hat{\mathbf{\Omega}})\,\mathrm{d}\hat{\mathbf{\Omega}}.$$

In the Fourier space we obtain

$$\begin{aligned}(1+\mathrm{i}\mathbf{k}\cdot\hat{\mathbf{\Omega}})\bar{\psi}(\mathbf{k},\,\hat{\mathbf{\Omega}})=&\,c\sum_{l=0}^{L}\sum_{m=-l}^{l}\frac{\omega_l^m}{\sqrt{4\pi(2l+1)}}\sqrt{\frac{(l+m)!}{(l-m)!}}\mathrm{e}^{-\mathrm{i}m\varphi_0}[\mathcal{R}_{\hat{\mathbf{k}}}Y_{lm}(\hat{\mathbf{\Omega}})]\bar{\psi}_l^m(\mathbf{k})\\&+c\frac{p(\hat{\mathbf{\Omega}},\,\hat{\mathbf{\Omega}}_0)}{1+\mathrm{i}\mathbf{k}\cdot\hat{\mathbf{\Omega}}_0}.\end{aligned}\tag{12}$$

This is expressed as

$$\begin{aligned}\bar{\psi}(\mathbf{k},\,\hat{\mathbf{\Omega}})=&\,\frac{c}{2}\frac{1}{1+\mathrm{i}\mathbf{k}\cdot\hat{\mathbf{\Omega}}}\sum_{l=0}^{L}\sum_{m=-l}^{l}\frac{\omega_l^m}{\sqrt{\pi(2l+1)}}\sqrt{\frac{(l+m)!}{(l-m)!}}\mathrm{e}^{-\mathrm{i}m\varphi_0}[\mathcal{R}_{\hat{\mathbf{k}}}Y_{lm}(\hat{\mathbf{\Omega}})]\bar{\psi}_l^m(\mathbf{k})\\&+c\frac{p(\hat{\mathbf{\Omega}},\,\hat{\mathbf{\Omega}}_0)}{(1+\mathrm{i}\mathbf{k}\cdot\hat{\mathbf{\Omega}})(1+\mathrm{i}\mathbf{k}\cdot\hat{\mathbf{\Omega}}_0)}.\end{aligned}\tag{13}$$

We obtain

$$\begin{aligned}\bar{\psi}_j^m(\mathbf{k})=&\,\Theta(L-|m|)c\sum_{l=|m|}^{L}\omega_l^mL_{jl}^m(z)\\&\times\left[\bar{\psi}_l^m(\mathbf{k})+\frac{1}{\sqrt{(2j+1)(2l+1)}}\mathrm{e}^{\mathrm{i}m\varphi_0}\mathcal{R}_{\hat{\mathbf{k}}}\frac{z}{z-\mu_0}P_l^m(\mu_0)\mathrm{e}^{-\mathrm{i}m\varphi_0}\right].\end{aligned}$$

For $|m| \leqslant j \leqslant L$, the above equation can be rewritten as

$$\sum_{l=|m|}^{L}[\delta_{jl} - c\omega_l^m L_{jl}^m(z)]\bar{\psi}_l^m(\mathbf{k}) = \sum_{l=|m|}^{L}\frac{c\omega_l^m L_{jl}^m(z)}{\sqrt{(2j+1)(2l+1)}}\frac{z}{z - \hat{\mathbf{k}}\cdot\hat{\mathbf{\Omega}}_0} \times P_l^m(\hat{\mathbf{k}}\cdot\hat{\mathbf{\Omega}}_0)\mathrm{e}^{\mathrm{i}m\varphi_0}\mathcal{R}_{\hat{\mathbf{k}}}\mathrm{e}^{-\mathrm{i}m\varphi_0}. \tag{14}$$

As is calculated in appendix C, an expression of $\psi(\mathbf{r}, \hat{\mathbf{\Omega}})$ similar to (2) is obtained using (13) and (14). In this case, the first term of $\psi(\mathbf{r}, \hat{\mathbf{\Omega}})$ becomes the once-collided term whereas the first term of (9) shows uncollided particles. In this section we continue as follows instead of performing the calculation in appendix C.

Let us look at (12). By using (1) we have

$$\mu Y_{lm}(\hat{\mathbf{\Omega}}) = \sqrt{\frac{(l+1)^2 - m^2}{4(l+1)^2 - 1}}Y_{l+1,m}(\hat{\mathbf{\Omega}}) + \sqrt{\frac{l^2 - m^2}{4l^2 - 1}}Y_{l-1,m}(\hat{\mathbf{\Omega}}).$$

Therefore

$$zh_l\bar{\psi}_l^m(\mathbf{k}) - (l+1-m)\bar{\psi}_{l+1}^m(\mathbf{k}) - (l+m)\bar{\psi}_{l-1}^m(\mathbf{k}) = zS_l^m(\mathbf{k}), \tag{15}$$

for $l > |m|$ and

$$zh_{|m|}\bar{\psi}_{|m|}^m(\mathbf{k}) - (|m|+1-m)\bar{\psi}_{|m|+1}^m(\mathbf{k}) = zS_{|m|}^m(\mathbf{k}),$$

where

$$h_l = 2l + 1 - \Theta(L-l)c\omega_l^m\frac{(l+m)!}{(l-m)!} = 2l + 1 - \Theta(L-l)c\beta_l$$

and

$$\begin{aligned} S_l^m(\mathbf{k}) &= \Theta(L-l)\frac{c\beta_l z}{z - \hat{\mathbf{k}}\cdot\hat{\mathbf{\Omega}}_0}\mathrm{e}^{\mathrm{i}m\varphi_0}[\mathcal{R}_{\hat{\mathbf{k}}}P_l^m(\mu_0)\mathrm{e}^{-\mathrm{i}m\varphi_0}] \\ &= \Theta(L-l)\frac{c\beta_l z}{z - \hat{\mathbf{k}}\cdot\hat{\mathbf{\Omega}}_0}\sum_{m'=-l}^{l}\mathrm{e}^{-\mathrm{i}m'\varphi_{\hat{\mathbf{k}}}}d_{m'm}^{\,l}(\theta_{\hat{\mathbf{k}}})P_l^{m'}(\mu_0)\mathrm{e}^{\mathrm{i}(m-m')\varphi_0}. \end{aligned}$$

We introduce Chandrasekhar polynomials of the first and second kinds as

$$\begin{aligned} &(l+m)g_{l-1}^m(z) - zh_l g_l^m(z) + (l+1-m)g_{l+1}^m(z) = 0, \\ &g_m^m(z) = \frac{(2m)!}{2^m m!} = (2m-1)!!, \quad g_{m+1}^m(z) = zh_m g_m^m(z) \end{aligned} \tag{16}$$

and

$$\begin{aligned} &(l+m)\rho_{l-1}^m(z) - zh_l\rho_l^m(z) + (l+1-m)\rho_{l+1}^m(z) = 0, \\ &\rho_m^m(z) = 0, \quad \rho_{m+1}^m(z) = \frac{z}{(2m-1)!!}, \end{aligned}$$

where $0 \leqslant m \leqslant l$. We note that

$$g_l^{-m}(z) = (-1)^m\frac{(l-m)!}{(l+m)!}g_l^m(z), \qquad \rho_l^{-m}(z) = (-1)^m\frac{(l-m)!}{(l+m)!}\rho_l^m(z).$$

We refer to [7] about how to numerically compute Chandrasekhar's polynomials of the first and second kinds. Davison used g_l^m to express the solution to the transport equation [4]. Numerical calculation of g_l^m was considered in [21]. Inönü pointed out that (16) has two linearly independent solutions, i.e., g_l^m and ρ_l^m [10]. We can express $\bar{\psi}_l^m$ as

$$\bar{\psi}_l^m = a^m(\mathbf{k})g_l^m(z) + b^m(\mathbf{k})\rho_l^m(z) + (1 - \delta_{l,|m|})z\sum_{j=|m|+1}^{l} \alpha_{l,j}^m(\mathbf{k})S_j^m(\mathbf{k}). \tag{17}$$

By setting $l = |m|$ in (17), we first notice that

$$a^m(\mathbf{k}) = \frac{2^m m!}{(2m)!}\bar{\psi}_m^m, \qquad a^{-m}(\mathbf{k}) = (-1)^m 2^m m! \bar{\psi}_m^{-m},$$

for $m \geqslant 0$. By plugging (17) we have

$$-(|m| + 1 - m)[b^m(\mathbf{k})\rho_{|m|+1}^m(z) + z\alpha_{|m|+1,|m|+1}^m(\mathbf{k})S_{|m|+1}^m(\mathbf{k})] = zS_{|m|}^m(\mathbf{k}).$$

Suppose $l > m$. Let us impose

$$zh_l\alpha_{l,j}^m - (l + 1 - m)\alpha_{l+1,j}^m - (l + m)\alpha_{l-1,j}^m = 0. \tag{18}$$

By substituting (17) for $\bar{\psi}_l^m$ in (15), we obtain

$$zh_l\alpha_{l,l}^m S_l^m - (l + 1 - m)(\alpha_{l+1,l}^m S_l^m + \alpha_{l+1,l+1}^m S_{l+1}^m) = S_l^m.$$

The left-hand side of the above equation can be rewritten as

$$\text{lhs} = -(l + 1 - m)\alpha_{l+1,l+1}^m S_{l+1}^m + (l + m)\alpha_{l-1,l}^m S_l^m.$$

Hence we can put

$$\alpha_{l-1,l}^m = \frac{1}{l + m}, \qquad \alpha_{l+1,l+1}^m = 0. \tag{19}$$

Thus we find

$$\begin{aligned} b^m(\mathbf{k}) &= -\,(2m - 1)\,!!\,S_m^m(\mathbf{k}), \\ b^{-m}(\mathbf{k}) &= -\,(-1)^m(2m)!(2m - 1)\,!!\,S_m^{-m}(\mathbf{k}), \end{aligned}$$

for $m \geqslant 0$. To find $\alpha_{l,j}^m(z)$, let us plug the expression $\alpha_{l,j}^m = u_j^m g_l^m + v_j^m \rho_l^m$ into $\alpha_{l,l}^m = 0$ and $\alpha_{l-1,l}^m = 1/(l + m)$. We obtain

$$\begin{aligned} u_l^m &= \frac{\rho_l^m}{(l + m)(g_{l-1}^m\rho_l^m - g_l^m\rho_{l-1}^m)}, \\ v_l^m &= \frac{-g_l^m}{(l + m)(g_{l-1}^m\rho_l^m - g_l^m\rho_{l-1}^m)}. \end{aligned}$$

Since we have the relation (appendix B)

$$(l + m)[g_{l-1}^m(z)\rho_l^m(z) - g_l^m(z)\rho_{l-1}^m(z)] = \frac{(l + m)!}{(l - m)!}\frac{z}{(2|m|)!},$$

we obtain

$$z\alpha_{l,j}^m(\mathbf{k}) = \frac{(l - m)!(2|m|)!}{(l + m)!}[\rho_j^m(z)g_l^m(z) - g_j^m(z)\rho_l^m(z)].$$

Finally for $-l \leqslant m \leqslant l$, (17) becomes

$$\bar{\psi}_l^m = \frac{g_l^m(z)}{g_{|m|}^m(z)}\bar{\psi}_{|m|}^m - z\frac{\rho_l^m(z)}{\rho_{|m|+1}^m(z)}S_{|m|}^m + (1 - \delta_{l,|m|})\frac{(l-m)!(2|m|)!}{(l+m)!} \times \sum_{j=|m|+1}^{l}[\rho_j^m(z)g_l^m(z) - g_j^m(z)\rho_l^m(z)]S_j^m = \frac{g_l^m(z)}{g_{|m|}^m(z)}\bar{\psi}_{|m|}^m - \frac{\chi_l^m}{g_{|m|}^m(z)}, \tag{20}$$

where

$$\chi_l^m(\mathbf{k}) = g_{|m|}^m(z)\left\{z\frac{\rho_l^m(z)}{\rho_{|m|+1}^m(z)}S_{|m|}^m(\mathbf{k}) - (1 - \delta_{l,|m|})\frac{(l-m)!(2|m|)!}{(l+m)!} \times \sum_{j=|m|+1}^{l}[\rho_j^m(z)g_l^m(z) - g_j^m(z)\rho_l^m(z)]S_j^m(\mathbf{k})\right\}. \tag{21}$$

We note that $\chi_{|m|}^m(\mathbf{k}) = 0$.

To find the initial term $\bar{\psi}_{|m|}^m$, we set $j = |m|$ in (8) and obtain

$$\sum_{l=|m|}^{L}[\delta_{|m|,l} - c\omega_l^m L_{|m|,l}^m(z)]\bar{\psi}_l^m = \frac{cz}{z - \hat{\mathbf{k}}\cdot\hat{\mathbf{\Omega}}_0}\sum_{l=|m|}^{L}\frac{\omega_l^m L_{|m|,l}^m(z)}{\sqrt{(2|m|+1)(2l+1)}} \times P_l^m(\hat{\mathbf{k}}\cdot\hat{\mathbf{\Omega}}_0)\mathrm{e}^{\mathrm{i}m\varphi_0}\mathcal{R}_{\hat{\mathbf{k}}}\mathrm{e}^{-\mathrm{i}m\varphi_0}.$$

For $|m| \leqslant L$, we obtain

$$\bar{\psi}_{|m|}^m = \frac{cz}{\Lambda^m(z)}\sum_{l=|m|}^{L}\omega_l^m Q_l^m(z)P_{|m|}^m(z) \times \left[\frac{1}{\sqrt{(2|m|+1)(2l+1)}}\frac{z}{z - \hat{\mathbf{k}}\cdot\hat{\mathbf{\Omega}}_0}P_l^m(\hat{\mathbf{k}}\cdot\hat{\mathbf{\Omega}}_0)\mathrm{e}^{\mathrm{i}m\varphi_0}\mathcal{R}_{\hat{\mathbf{k}}}\mathrm{e}^{-\mathrm{i}m\varphi_0} - \frac{\chi_l^m(\mathbf{k})}{g_{|m|}^m(z)}\right], \tag{22}$$

where (appendix B)

$$\Lambda^m(z) = 1 - cz\sum_{l=|m|}^{L}\omega_l^m Q_l^m(z)P_{|m|}^m(z)\frac{g_l^m(z)}{g_{|m|}^m(z)} = \frac{(L+1-m)!}{(L+m)!}\frac{P_{|m|}^m(z)}{g_{|m|}^m(z)}[g_{L+1}^m(z)Q_L^m(z) - g_L^m(z)Q_{L+1}^m(z)].$$

We can calculate $\bar{\psi}_l^m(\mathbf{k})$ using (20) and (22).

We note that the Fourier transform of the angular flux is given by

$$\bar{\psi}(\mathbf{k},\hat{\mathbf{\Omega}}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\mathrm{e}^{-\mathrm{i}m\varphi_0}\bar{\psi}_l^m(\mathbf{k})\mathcal{R}_{\hat{\mathbf{k}}}Y_{lm}(\hat{\mathbf{\Omega}}).$$

Using (20) the above equation is rewritten as

$$\bar{\psi}(\mathbf{k}, \hat{\mathbf{\Omega}}) = \sum_{m=-\infty}^{\infty} [\phi_{\hat{\mathbf{k}}}^{m}(z, \hat{\mathbf{\Omega}})\bar{\psi}_{|m|}^{m}(z, \hat{\mathbf{k}}) - T_{\hat{\mathbf{k}}}^{m}(z, \hat{\mathbf{\Omega}})],$$

where

$$\phi_{\hat{\mathbf{k}}}^{m}(z, \hat{\mathbf{\Omega}}) = \frac{1}{g_{|m|}^{m}(z)} \sum_{l=|m|}^{\infty} \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} \mathrm{e}^{-\mathrm{i}m\varphi_0} g_l^m(z) \mathcal{R}_{\hat{\mathbf{k}}} Y_{lm}(\hat{\mathbf{\Omega}}),$$
$$T_{\hat{\mathbf{k}}}^{m}(z, \hat{\mathbf{\Omega}}) = \frac{1}{g_{|m|}^{m}(z)} \sum_{l=|m|}^{\infty} \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} \mathrm{e}^{-\mathrm{i}m\varphi_0} \chi_l^m(\mathbf{k}) \mathcal{R}_{\hat{\mathbf{k}}} Y_{lm}(\hat{\mathbf{\Omega}}).$$

Note that the dependence of $\bar{\psi}_m^m$ in (22) on $\mathbf{k}$ is split into z and $\hat{\mathbf{k}}$. The angular flux is then given by

$$\begin{aligned}\psi(\mathbf{r}, \hat{\mathbf{\Omega}}) &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\bar{\psi}(\mathbf{k}, \hat{\mathbf{\Omega}})\,\mathrm{d}\mathbf{k}\\ &= \frac{1}{(2\pi)^3}\int_{\mathbb{S}^2}\int_0^{\infty} k^2 \mathrm{e}^{\mathrm{i}kr\hat{\mathbf{k}}\cdot\hat{\mathbf{r}}} \sum_{m=-\infty}^{\infty} [\phi_{\hat{\mathbf{k}}}^{m}(\mathrm{i}/k, \hat{\mathbf{\Omega}})\bar{\psi}_{|m|}^{m}(\mathrm{i}/k, \hat{\mathbf{k}}) - T_{\hat{\mathbf{k}}}^{m}(\mathrm{i}/k, \hat{\mathbf{\Omega}})]\,\mathrm{d}k\mathrm{d}\hat{\mathbf{k}}.\end{aligned} \tag{23}$$

By explicitly writing (23), we obtain (3).

We have

$$\begin{aligned}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}kr\hat{\mathbf{k}}\cdot\hat{\mathbf{r}}}\mathrm{e}^{-\mathrm{i}m'\varphi_{\hat{\mathbf{k}}}}\,\mathrm{d}\varphi_{\hat{\mathbf{k}}} &= \int_0^{2\pi} \mathrm{e}^{\mathrm{i}kr\sin\theta_{\hat{\mathbf{k}}}\sin\theta_{\hat{\mathbf{r}}}\cos(\varphi_{\hat{\mathbf{k}}}-\varphi_{\hat{\mathbf{r}}})}\mathrm{e}^{\mathrm{i}kr\cos\theta_{\hat{\mathbf{k}}}\cos\theta_{\hat{\mathbf{r}}}}\mathrm{e}^{-\mathrm{i}m'\varphi_{\hat{\mathbf{k}}}}\,\mathrm{d}\varphi_{\hat{\mathbf{k}}},\\ &= \mathrm{e}^{\mathrm{i}kr\cos\theta_{\hat{\mathbf{k}}}\cos\theta_{\hat{\mathbf{r}}}}\mathrm{e}^{-\mathrm{i}m'\varphi_{\hat{\mathbf{r}}}}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}kr\sin\theta_{\hat{\mathbf{k}}}\sin\theta_{\hat{\mathbf{r}}}\cos\varphi_{\hat{\mathbf{k}}}}\mathrm{e}^{-\mathrm{i}m'\varphi_{\hat{\mathbf{k}}}}\,\mathrm{d}\varphi_{\hat{\mathbf{k}}},\\ &= 2\pi\mathrm{i}^{m'}J_{m'}(kr\sin\theta_{\hat{\mathbf{k}}}\sin\theta_{\hat{\mathbf{r}}})\mathrm{e}^{\mathrm{i}kr\cos\theta_{\hat{\mathbf{k}}}\cos\theta_{\hat{\mathbf{r}}}}\mathrm{e}^{-\mathrm{i}m'\varphi_{\hat{\mathbf{r}}}},\end{aligned}$$

where we noted the Hansen-Bessel formula:

$$J_m(x) = \frac{1}{2\pi\mathrm{i}^m}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}x\cos\varphi}\mathrm{e}^{-\mathrm{i}m\varphi}\,\mathrm{d}\varphi. \tag{24}$$

Hence we have (4) if $\bar{\psi}_{|m|}^{m}$ does not depend on $\varphi_{\hat{\mathbf{k}}}$.

5. Energy density

We consider the energy density $u(\mathbf{r})$ using (4). Let us assume an isotropic source $\delta(\mathbf{r})$. Without loss of generality we can set

$$\varphi_{\hat{\mathbf{r}}} = 0, \qquad \theta_{\hat{\mathbf{r}}} = 0.$$

We normalize $u(\mathbf{r})$ with the speed of neutrons. We compute $u(\mathbf{r})$ as

$$\begin{aligned}u(\mathbf{r}) &= \int_{\mathbb{S}^2}\int_{\mathbb{S}^2} G(\mathbf{r}, \hat{\mathbf{\Omega}}; \hat{\mathbf{\Omega}}_0)\,\mathrm{d}\hat{\mathbf{\Omega}}\mathrm{d}\hat{\mathbf{\Omega}}_0\\ &= \frac{1}{r^2}\mathrm{e}^{-r} + \frac{1}{(2\pi)^2}\int_0^{\infty} k^2 \int_{-1}^{1} \mathrm{e}^{\mathrm{i}kr\mu_{\hat{\mathbf{k}}}} \int_{\mathbb{S}^2} \kappa_{00}(\mathbf{k})\,\mathrm{d}\hat{\mathbf{\Omega}}_0\,\mathrm{d}\mu_{\hat{\mathbf{k}}}\mathrm{d}k.\end{aligned} \tag{25}$$

We note that $\kappa_{00}(\mathbf{k})$ is obtained as

$$\kappa_{00}(\mathbf{k}) = \left\{(L+1)\left[g_{L+1}^0\left(\frac{\mathrm{i}}{k}\right)Q_L^0\left(\frac{\mathrm{i}}{k}\right) - g_L^0\left(\frac{\mathrm{i}}{k}\right)Q_{L+1}^0\left(\frac{\mathrm{i}}{k}\right)\right]\right\}^{-1} \frac{c\frac{\mathrm{i}}{k}Q_0\left(\frac{\mathrm{i}}{k}\right)}{1+\mathrm{i}\mathbf{k}\cdot\hat{\mathbf{\Omega}}_0},$$

where

$$Q_0\left(\frac{\mathrm{i}}{k}\right) = \frac{1}{2}\ln\frac{\mathrm{i}/k+1}{\mathrm{i}/k-1} = -\mathrm{i}\tan^{-1}(k). \tag{26}$$

Noting that

$$\int_{\mathbb{S}^2} \frac{1}{1+\mathrm{i}\mathbf{k}\cdot\hat{\mathbf{\Omega}}_0}\,\mathrm{d}\hat{\mathbf{\Omega}}_0 = 4\pi\frac{\tan^{-1}(k)}{k},$$

we obtain

$$u(\mathbf{r}) = \frac{1}{r^2}\mathrm{e}^{-r} + \frac{c}{\pi(L+1)}\int_0^\infty \frac{2\sin(kr)[\tan^{-1}(k)]^2}{kr\left[g_{L+1}^0\left(\frac{\mathrm{i}}{k}\right)Q_L\left(\frac{\mathrm{i}}{k}\right) - g_L^0\left(\frac{\mathrm{i}}{k}\right)Q_{L+1}\left(\frac{\mathrm{i}}{k}\right)\right]}\,\mathrm{d}k.$$

In particular if $L=0$ (isotropic scattering), we have

$$u(\mathbf{r}) = \frac{1}{r^2}\mathrm{e}^{-r} + \frac{2c}{\pi}\int_0^\infty \frac{\sin(kr)[\tan^{-1}(k)]^2}{kr\left[1-\frac{c}{k}\tan^{-1}(k)\right]}\,\mathrm{d}k.$$

This is the expression obtained from the textbook Fourier transform approach shown in appendix A.

6. Concluding remarks

When aiming at benchmarking [6], the two formulas (2) and (3) derived in the present paper are useful because solutions to the three-dimensional monoenergetic neutron transport equation in an infinite medium with anisotropic scattering are analytically obtained. The solutions do not suffer from statistical errors unlike Monte Carlo simulation.

We have obtained the angular flux for anisotropic scattering by means of the Fourier transform. The calculation developed here can be compared to the method of rotated reference frames [15, 17], which uses the Fourier transform and rotated reference frames. It is known that the method of rotated reference frames has instability when spherical harmonics with large degrees are used [16]. In the present formulation, decomposition of the angular flux into eigenmodes is not introduced. In this way, such instability does not arise.

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The main results of this paper were obtained by extending the one-dimensional Fourier transform calculation by B D Ganapol. The author learned this calculation at the 24th International Conference on Transport Theory (ICTT24) held in Taormina, Sicily, Italy in September 2015.

Appendix A. The case of isotropic scattering

In the isotropic case we can obtain $G(\mathbf{r}, \hat{\mathbf{\Omega}}; \hat{\mathbf{\Omega}}_0)$ as follows [1, 11]. The Green's function obeys

$$\hat{\mathbf{\Omega}} \cdot \nabla G(\mathbf{r}, \hat{\mathbf{\Omega}}; \hat{\mathbf{\Omega}}_0) + G(\mathbf{r}, \hat{\mathbf{\Omega}}; \hat{\mathbf{\Omega}}_0) = \frac{c}{4\pi} \int_{\mathbb{S}^2} G(\mathbf{r}, \hat{\mathbf{\Omega}}; \hat{\mathbf{\Omega}}_0)\, \mathrm{d}\hat{\mathbf{\Omega}} + \delta(\mathbf{r})\delta(\hat{\mathbf{\Omega}} - \hat{\mathbf{\Omega}}_0).$$

We write G in terms of G_0 as

$$\begin{aligned} G(\mathbf{r}, \hat{\mathbf{\Omega}}; \hat{\mathbf{\Omega}}_0) = &\int_{\mathbb{R}^3} \int_{\mathbb{S}^2} G_0(\mathbf{r} - \mathbf{r}', \hat{\mathbf{\Omega}}; \hat{\mathbf{\Omega}}') \\ &\times \left\{ \frac{c}{4\pi} \int_{\mathbb{S}^2} G(\mathbf{r}', \hat{\mathbf{\Omega}}''; \hat{\mathbf{\Omega}}_0)\, \mathrm{d}\hat{\mathbf{\Omega}}'' + \delta(\mathbf{r}')\delta(\hat{\mathbf{\Omega}}' - \hat{\mathbf{\Omega}}_0) \right\} \mathrm{d}\mathbf{r}' \mathrm{d}\hat{\mathbf{\Omega}}'. \end{aligned}$$

We define

$$\begin{aligned} U(\mathbf{r}; \hat{\mathbf{\Omega}}_0) &= \int_{\mathbb{S}^2} G(\mathbf{r}, \hat{\mathbf{\Omega}}''; \hat{\mathbf{\Omega}}_0)\, \mathrm{d}\hat{\mathbf{\Omega}}'' \\ &= \int_{\mathbb{R}^3} \frac{1}{|\mathbf{r} - \mathbf{r}'|^2} \mathrm{e}^{-|\mathbf{r}-\mathbf{r}'|} \left\{ \frac{c}{4\pi} U(\mathbf{r}'; \hat{\mathbf{\Omega}}_0) + \delta(\mathbf{r}')\delta\left(\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} - \hat{\mathbf{\Omega}}_0 \right) \right\} \mathrm{d}\mathbf{r}'. \end{aligned}$$

Since the Fourier transform is obtained as

$$\bar{U}(\mathbf{k}) = \frac{1}{1 + \mathrm{i}\mathbf{k} \cdot \hat{\mathbf{\Omega}}_0} \left[1 - \frac{c}{k} \tan^{-1}(k) \right]^{-1},$$

we obtain

$$U(\mathbf{r}; \hat{\mathbf{\Omega}}_0) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}}{1 + \mathrm{i}\mathbf{k} \cdot \hat{\mathbf{\Omega}}_0} \left[1 - \frac{c}{k} \tan^{-1}(k) \right]^{-1} \mathrm{d}\mathbf{k}.$$

Finally, the Green's function is written as

$$\begin{aligned} G(\mathbf{r}, \hat{\mathbf{\Omega}}; \hat{\mathbf{\Omega}}_0) = G_0(\mathbf{r}, \hat{\mathbf{\Omega}}; \hat{\mathbf{\Omega}}_0) &+ \frac{c}{4\pi(2\pi)^3} \int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} \\ &\times \frac{1}{(1 + \mathrm{i}\mathbf{k} \cdot \hat{\mathbf{\Omega}})(1 + \mathrm{i}\mathbf{k} \cdot \hat{\mathbf{\Omega}}_0)} \left[1 - \frac{c}{k} \tan^{-1}(k) \right]^{-1} \mathrm{d}\mathbf{k}. \end{aligned}$$

Let us consider the energy density for the isotropic source $\delta(\mathbf{r})$. We obtain

$$\begin{aligned} u(\mathbf{r}) &= \int_{\mathbb{S}^2 \times \mathbb{S}^2} G(\mathbf{r}, \hat{\mathbf{\Omega}}; \hat{\mathbf{\Omega}}')\, \mathrm{d}\hat{\mathbf{\Omega}} \mathrm{d}\hat{\mathbf{\Omega}}' \\ &= \int_{\mathbb{S}^2 \times \mathbb{S}^2} G_0(\mathbf{r}, \hat{\mathbf{\Omega}}; \hat{\mathbf{\Omega}}')\, \mathrm{d}\hat{\mathbf{\Omega}} \mathrm{d}\hat{\mathbf{\Omega}}' + \frac{c}{4\pi(2\pi)^3} \int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} \\ &\quad \times \int_{\mathbb{S} \times \mathbb{S}^2} \frac{1}{(1 + \mathrm{i}\mathbf{k} \cdot \hat{\mathbf{\Omega}})(1 + \mathrm{i}\mathbf{k} \cdot \hat{\mathbf{\Omega}}')} \mathrm{d}\hat{\mathbf{\Omega}} \mathrm{d}\hat{\mathbf{\Omega}}' \\ &\quad \times \left[1 - \frac{c}{k} \tan^{-1}(k) \right]^{-1} \mathrm{d}\mathbf{k} \\ &= \frac{\mathrm{e}^{-r}}{r^2} + \frac{2c}{\pi} \int_0^\infty \frac{\sin(Ks)}{r} \frac{[\tan^{-1}(k)]^2}{k - c \tan^{-1}(k)} \mathrm{d}k. \end{aligned}$$

Appendix B. Christoffel–Darboux formulas

We consider the following two recurrence relations [8].

$$z a_l q_l^m(z) - (l+1-m) q_{l+1}^m(z) - (l+m) q_{l-1}^m(z) = 0, \tag{B.1}$$

$$\mu b_l r_l^m(\mu) - (l+1-m) r_{l+1}^m(\mu) - (l+m) r_{l-1}^m(\mu) = 0. \tag{B.2}$$

By subtracting $q_l^m(z)[(l-m)!/(l+m)!] \times$ (B.2) from $r_l^m(\mu)[(l-m)!/(l+m)!] \times$ (B.1) we obtain [9]

$$(z a_l - \mu b_l)\frac{(l-m)!}{(l+m)!} q_l^m(z) r_l^m(z) + t_{l+1}^m(z, \mu) - t_l^m(z, \mu) = 0, \tag{B.3}$$

where

$$t_l^m(z, \mu) = \frac{(l-m)!}{(l-1+m)!}[q_{l-1}^m(z) r_l^m(\mu) - q_l^m(z) r_{l-1}^m(\mu)].$$

Suppose $l_0 > |m| + 1$. By taking the summation $\sum_{l=|m|+1}^{l_0}$ we obtain

$$\begin{aligned}
&\frac{(l_0+1-m)!}{(l_0+m)!}[q_{l_0}^m(z) r_{l_0+1}^m(\mu) - q_{l_0+1}^m(z) r_{l_0}^m(\mu)] \\
&\quad = \sum_{l=|m|+1}^{l_0} (\mu b_l - z a_l)\frac{(l-m)!}{(l+m)!} q_l^m(z) r_l^m(\mu) \\
&\qquad + \frac{(|m|+1-m)!}{(|m|+m)!}[q_{|m|}^m(z) r_{|m|+1}^m(\mu) - q_{|m|+1}^m(z) r_{|m|}^m(\mu)].
\end{aligned}$$

If we set

$$l_0 = l - 1, \quad a_l = b_l = h_l, \quad q_l^m = g_l^m, \quad r_l^m = \rho_l^m,$$

we obtain

$$(l+m)[g_{l-1}^m(z) \rho_l^m(z) - g_l^m(z) \rho_{l-1}^m(z)] = \frac{(l+m)!}{(l-m)!}\frac{z}{(2|m|)!}.$$

If we set

$$l_0 = L, \quad a_l = h_l, \quad b_l = 2l + 1, \quad q_l^m = \frac{g_l^m}{g_{|m|}^m}, \quad r_l^m = Q_l^m P_{|m|}^m,$$

we obtain

$$\begin{aligned}
&\frac{(L+1-m)!}{(L+m)!}\frac{P_{|m|}^m(z)}{g_{|m|}^m(z)}[g_{L+1}^m(z) Q_L^m(z) - g_L^m(z) Q_{L+1}^m(z)] \\
&\quad = -cz \sum_{l=|m|+1}^{L} \omega_l^m g_l^m(z) Q_l^m(z) \\
&\qquad - \frac{(|m|+1-m)!}{(|m|+m)!}\frac{P_{|m|}^m(z)}{g_{|m|}^m(z)}[g_{|m|}^m(z) Q_{|m|+1}^m(z) - g_{|m|+1}^m(z) Q_{|m|}^m(z)].
\end{aligned}$$

Therefore we have

$$\frac{(L+1-m)!}{(L+m)!}\frac{P_{|m|}^{m}(z)}{g_{|m|}^{m}(z)}[g_{L+1}^{m}(z)Q_{L}^{m}(z)-g_{L}^{m}(z)Q_{L+1}^{m}(z)]$$
$$=-cz\sum_{l=|m|}^{L}\omega_{l}^{m}\frac{g_{l}^{m}(z)}{g_{|m|}^{m}(z)}Q_{l}^{m}(z)P_{|m|}^{m}(z)+1.$$

Here we used

$$Q_{|m|+1}^{m}(z)=(2|m|+1)zQ_{|m|}^{m}(z)-\frac{[(2|m|)!]^{\mathrm{sgn}(m)}}{P_{|m|}^{m}(z)}.$$

The above relation is derived from (6).

Appendix C. The Fourier transform with ballistic subtraction

We additionally introduce the following matrices and vectors.

$$\{\mathbf{S}\}_{jl}=\frac{1}{\sqrt{2j+1}}\delta_{jl},$$
$$\{\bar{\boldsymbol{\psi}}^{m}(\mathbf{k})\}_{l}=\bar{\psi}_{l}^{m}(\mathbf{k}).$$

Equation (14) can be expressed as

$$[\mathbf{I}-c\mathbf{L}^{m}(z)\mathbf{W}^{m}]\bar{\boldsymbol{\psi}}^{m}(\mathbf{k})=c\mathbf{S}\mathbf{L}^{m}(z)\mathbf{S}\mathbf{W}^{m}(z)\frac{z}{z-\hat{\mathbf{k}}\cdot\hat{\boldsymbol{\Omega}}_{0}}\mathbf{P}^{m}(\hat{\mathbf{k}},\hat{\boldsymbol{\Omega}}_{0}).$$

Using (13), $\bar{\boldsymbol{\psi}}^{m}(\mathbf{k})$ is obtained as

$$\bar{\psi}(\mathbf{k},\hat{\boldsymbol{\Omega}})=\frac{cp(\hat{\boldsymbol{\Omega}},\hat{\boldsymbol{\Omega}}_{0})}{(1+\mathrm{i}\mathbf{k}\cdot\hat{\boldsymbol{\Omega}})(1+\mathrm{i}\mathbf{k}\cdot\hat{\boldsymbol{\Omega}}_{0})}+\frac{c}{4\pi}\frac{z}{z-\hat{\mathbf{k}}\cdot\hat{\boldsymbol{\Omega}}}\mathbf{P}^{m}(\hat{\mathbf{k}},\hat{\boldsymbol{\Omega}})^{\dagger}\mathbf{W}^{m}\bar{\boldsymbol{\psi}}^{m}(\mathbf{k}).$$

Therefore we obtain

$$\psi(\mathbf{r},\hat{\boldsymbol{\Omega}})=cp(\hat{\boldsymbol{\Omega}},\hat{\boldsymbol{\Omega}}_{0})\int_{0}^{\infty}\int_{0}^{\infty}\mathrm{e}^{-r_1}\mathrm{e}^{-r_2}\delta(\mathbf{r}-r_1\hat{\boldsymbol{\Omega}}-r_2\hat{\boldsymbol{\Omega}}_{0})\,\mathrm{d}r_1\mathrm{d}r_2$$
$$+\frac{c^2}{2(2\pi)^4}\int_{\mathbb{R}^3}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\frac{z}{z-\hat{\mathbf{k}}\cdot\hat{\boldsymbol{\Omega}}}\frac{z}{z-\hat{\mathbf{k}}\cdot\hat{\boldsymbol{\Omega}}_{0}}$$
$$\times\mathbf{P}^{m}(\hat{\mathbf{k}},\hat{\boldsymbol{\Omega}})^{\dagger}\mathbf{W}^{m}[\mathbf{I}-c\mathbf{L}^{m}\mathbf{W}^{m}]^{-1}\mathbf{S}\mathbf{L}^{m}(z)\mathbf{S}\mathbf{W}^{m}(z)\mathbf{P}^{m}(\hat{\mathbf{k}},\hat{\boldsymbol{\Omega}}_{0})\,\mathrm{d}\mathbf{k}.$$

We note that the first term in the above equation is the once-collided term in the collision expansion:

$$(\text{once}-\text{collided term})=cp(\hat{\boldsymbol{\Omega}},\hat{\boldsymbol{\Omega}}_{0})\Theta(\pi-\tau-\tau_0)\delta(|\varphi-\varphi_0|-\pi)\frac{\mathrm{e}^{-r(\sin\tau+\sin\tau_0)/\sin(\tau+\tau_0)}}{r\sin\tau\sin\tau_0},$$

where $\cos\tau=\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\Omega}}$, $\cos\tau_0=\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\Omega}}_0$.

References

[1] Case K M and Zweifel P F 1967 *Linear Transport Theory* (Reading, MA: Addison-Wesley)
[2] Chandrasekhar S 1960 *Radiative Transfer* (New York: Dover)
[3] Dede K M 1964 An explicit solution of the one velocity multi-dimensional Boltzmann-equation in P_N approximation *Nukleonik* **6** 267–71

[4] Davison B 1957 *Neutron Transport Theory* (Oxford: Oxford University Press)
[5] Gonapol B D 2000 A consistent theory of neutral particle transport in an infinite medium *Transp. Theory Stat. Phys.* **29** 43–68
[6] Gonapol B D 2008 *Analytical Benchmarks for Nuclear Engineering Applications* (*Case Study in Neutron Transport Theory*) (Paris: OECD Publishing)
[7] Gonapol B D 2014 Chandrasekhar polynomials and the solution to the transport equation in an infinite medium *J. Comput. Theor. Trans.* **43** 433–73
[8] Ganapol B D 2015 The infinite medium Green's function of monoenergetic neutron transport theory via Fourier transform *Nucl. Sci. Eng.* **180** 224–46
[9] Garcia R D M and Siewert C E 1982 On the dispersion function in particle transport theory *J. Appl. Math. Phys.* **33** 801–6
[10] Inönü E 1970 Orthogonality of a set of polynomials encountered in neutron-transport and radiative-transfer theories *J. Math. Phys.* **11** 568–77
[11] Ishimaru A 1978 *Wave Propagation and Scattering in Random Media* (New York: Academic)
[12] Kobayashi K 1977 Spherical harmonics solutions of multi-dimensional neutron transport equation by finite Fourier transformation *J. Nucl. Sci. Technol.* **14** 489–501
[13] Machida M 2014 Singular eigenfunctions for the three-dimensional radiative transport equation *J. Opt. Soc. Am.* A **31** 67–74
[14] Machida M 2015 An F_N method for the radiative transport equation in three-dimensions *J. Phys. A: Math. Theor.* **48** 325001
[15] Markel V A 2004 Modified spherical harmonics method for solving the radiative transport equation *Waves Random Media* **14** L13–9
[16] Markel V A 2012 private communication
[17] Panasyuk G, Schotland J C and Markel V A 2006 Radiative transport equation in rotated reference frames *J. Phys. A: Math. Gen.* **39** 115–37
[18] Schotland J C and Markel V A 2007 Fourier–Laplace structure of the inverse scattering problem for the radiative transport equation *Inverse Problem Imaging* **1** 181–8
[19] Siewert C E 1978 The F_N method for solving radiative-transfer problems in plane geometry *Astrophys. Space Sci.* **58** 131–7
[20] Siewert C E and Benoist P 1979 The F_N method in neutron-transport theory. Part I: Theory and applications *Nucl. Sci. Eng.* **69** 156–60
[21] Siewert C E and Garcia R D M 1990 On computing the Chandrasekhar polynomials in high order and high degree *J. Quant. Spectrosc. Radiat. Transfer* **43** 201–5
[22] Varshalovich D A, Moskalev A N and Khersonskii V K 1988 *Quantum Theory of Angular Momentum* (Singapore: World Scientific)