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Inverse Born series for the radiative transport equation

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Abstract

We propose a direct reconstruction method for the inverse transport problem that is based on inversion of the Born series. We characterize the approximation error of the method and illustrate its use in numerical simulations.

Keywords: radiative transport, inverse problem, optical tomography

(Some figures may appear in colour only in the online journal)

1. Introduction

Optical tomography is a biomedical imaging modality that uses multiply-scattered light to probe the spatial structure of biological tissue [1, 2]. The corresponding inverse problem is to reconstruct the optical properties of a highly-scattering medium from boundary measurements. The mathematical formulation of this problem is dictated primarily by spatial scale, ranging from radiative transport theory at small scales to diffusion theory at the macroscale [2]. In either case, the inverse problem is both nonlinear and severely ill-posed, which has the effect of limiting the resolution of reconstructed images.

In radiative transport theory the propagation of multiply-scattered light through a material medium is formulated in terms of a conservation law that accounts for gains and losses of electromagnetic energy due to scattering and absorption. Let Ω be a bounded domain in $\mathbb{R}^d$, for $d \geqslant 2$, with a smooth boundary $\partial\Omega$. The fundamental quantity of interest is the specific intensity $u(x, \theta)$, which is the intensity of light at the point $x \in \Omega$ in the direction $\theta \in \mathbb{S}^{d-1}$. The specific intensity obeys the radiative transport equation (RTE)

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$$\theta \cdot \nabla u(x, \theta) + \sigma(x)u(x, \theta) = \int_{\mathbb{S}^{d-1}} k\left(\theta, \theta'\right)u\left(x, \theta'\right)\mathrm{d}\theta', \quad (x, \theta) \in \Omega \times \mathbb{S}^{d-1}, \tag{1}$$

$$u(x, \theta) = g(x, \theta), \quad (x, \theta) \in \Gamma_-, \tag{2}$$

which we have written in its time-independent form. The attenuation coefficient $\sigma(x)$ is assumed to be nonnegative for all $x \in \Omega$. In addition, the scattering kernel $k(\theta, \theta')$ is nonnegative and obeys the reciprocity relation $k(\theta, \theta') = k(-\theta', -\theta)$ and is normalized so that

$$\int_{\mathbb{S}^{d-1}} k\left(\theta, \theta'\right)\mathrm{d}\theta' = 1, \quad \theta \in \mathbb{S}^{d-1}. \tag{3}$$

We also introduce the sets $\Gamma_\pm$ which are defined by

$$\Gamma_\pm = \left\{ (x, \theta) \in \partial\Omega \times \mathbb{S}^{d-1} : \pm\theta \cdot n(x) > 0\right\}, \tag{4}$$

with n being the outer unit normal to $\partial\Omega$.

The inverse transport problem is to recover the coefficient σ from knowledge of the albedo operator $\Lambda_\sigma : g \mapsto u\,|_{\Gamma_+}$, where we assume that the scattering kernel k is known. We note that in physical terms $u\,|_{\Gamma_+}$, corresponds to outgoing measurements of the specific intensity. There is a considerable body of work on the inverse transport problem [4, 10–12, 21, 22, 25–30]. If Λ_σ is known, then σ can be reconstructed uniquely. This result follows from analyzing the singularities of the albedo operator. The singular structure can be used to recover σ with good stability.

It is important to note that angularly-resolved measurements of the specific intensity are difficult to obtain in practice. Thus only partial knowledge of Λ_σ is available from experiments. The inverse problem with angularly-averaged measurements has been analyzed in [5, 6, 16, 17]. It can be seen that the effect of angular averaging is to destroy the singularities that are present in the albedo operator. This results in severe ill-posedness of the inverse problem. In particular, it is possible to reconstruct the low-frequency part of σ with logarithmic stability.

In previous work, we have proposed a direct method to solve the inverse problem of optical tomography that is based on inversion of the Born series [15, 19, 20]. In this approach, the solution to the inverse problem is expressed as an explicitly computable functional of the scattering data. In combination with a spectral method for solving the linear inverse problem, the inverse Born series leads to an image reconstruction algorithm with analyzable error and stability. The inverse Born series has also been applied to the Calderon problem [3].

To date, the inverse Born series has only been employed to study the simplest inverse problem in optical tomography, namely the reconstruction of the attenuation coefficient within the diffusion approximation (DA) to the RTE. While this is an important first step, the DA breaks down in many situations of practical interest, including those of relatively weak scattering and strong absorption. Moreover, use of the DA limits the resolution of reconstructed images to macroscopic scales.

In this paper, we apply the inverse Born series methodology to the inverse transport problem. We characterize the approximation error of the method and illustrate its use in numerical simulations. We find that the series appears to converge quite rapidly for low-contrast objects. As the contrast is increased, the higher order terms systematically improve the reconstructions until, at sufficiently large contrast, the series diverges.

The remainder of this paper is organized as follows. In section 2, we construct the Born series for the RTE. We then derive various estimates that are later used to study the convergence of the inverse series. The inversion of the Born series is taken up in section 3, where we also obtain our main results on the convergence and approximation error of the method. We consider the Born series and inverse Born series for angularly-averaged measurements in

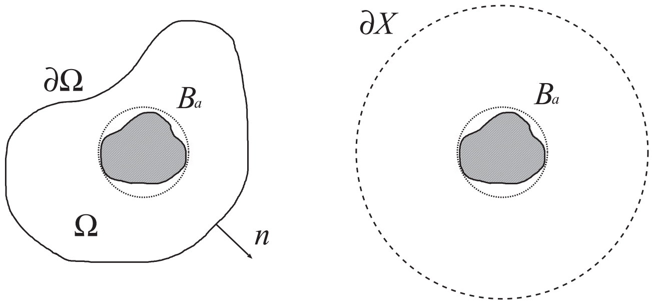

Figure 1. Illustrating the inverse transport problem in bounded (left) and unbounded domains (right).

section 4. In section 5, we study in detail the case of a slab-shaped medium and present the results of numerical reconstructions. Our conclusions are presented in section 6. Some details concerning the slab problem and singular eigenfunctions for the RTE are described in the appendices.

2. Forward problem

We consider the RTE (1) in a bounded domain. The forward problem is to determine the specific intensity u for a given attenuation σ. We assume that the attenuation coefficient σ is of the form

$$\sigma(x) = \sigma_0(1 + \eta(x)), \tag{5}$$

where the background attenuation $\sigma_0 = \sigma\,|_{\partial\Omega}$ is constant and $\eta(x) > -1$ for all $x \in \Omega$. The function η is the spatially varying part of the attenuation coefficient; it is assumed to be supported in a closed ball B_a of radius a, centered at the origin, as shown in figure 1 (left). The specific intensity u obeys the integral equation

$$u(x, \theta) = u_0(x, \theta) - \sigma_0 \int_{\Omega\times\mathbb{S}^{d-1}} G\left(x, \theta; x', \theta'\right)\eta(x')u\left(x', \theta'\right)\mathrm{d}x'\mathrm{d}\theta',\\ (x, \theta) \in \Omega \times \mathbb{S}^{d-1}. \tag{6}$$

Here u_0 obeys (1) with $\eta = 0$ and G is the Green's function for the background medium, which satisfies the equation

$$\begin{aligned} \nabla_x G\left(x, \theta; x', \theta'\right) + \sigma_0 G\left(x, \theta; x', \theta'\right) = \int_{\mathbb{S}^{d-1}} & k\left(\theta, \theta''\right) \times G\left(x, \theta''; x', \theta'\right)\mathrm{d}\theta'' \\ & + \delta(x - x')\delta\left(\theta - \theta'\right), \\ & G\left(x, \theta; x', \theta'\right) = 0, \quad (x, \theta) \in \Gamma_-. \end{aligned} \tag{7}$$

It is easily seen that u_0 is given by the formula

$$u_0(x, \theta) = \int_{\partial\Omega}\int_{\theta'\cdot n<0} G\left(x, \theta; x', \theta'\right)|\theta' \cdot n(x')|\, g\left(x', \theta'\right)\mathrm{d}x'\mathrm{d}\theta'. \tag{8}$$

We note that (8) should not be confused with the collision expansion for the RTE [9].

The integral equation (6) has a unique solution. If we apply fixed point iteration to (6), beginning with $u = u_0$, we obtain an infinite series for u of the form

$$u(x, \theta) = u_0(x, \theta) + u_1(x, \theta) + u_2(x, \theta) + \cdots, \tag{9}$$

where

$$u_{j+1}(x, \theta) = -\sigma_0 \int_{\Omega\times\mathbb{S}^{d-1}} G\left(x, \theta; x', \theta'\right)u_j\left(x', \theta'\right)\eta(x')\mathrm{d}x'\mathrm{d}\theta', \quad j = 0, 1, \ldots. \tag{10}$$

We will refer to (9) as the Born series and the approximation to u that results from retaining only the linear term in the series as the first Born approximation. It will prove useful to express the Born series as a formal power series in tensor powers of η of the form

$$\Phi = K_1\eta + K_2\eta \otimes \eta + K_3\eta \otimes \eta \otimes \eta + \cdots, \tag{11}$$

where the scattering data $\Phi = u_0 - u$. The forward operators K_j are defined as

$$\begin{aligned}\left(K_j f\right)(x, \theta) = (-1)^{j+1}\sigma_0^j \int_{\Gamma_a\times\cdots\times\Gamma_a} & G\left(x, \theta; x_1', \theta_1'\right)G\left(x_1', \theta_1'; x_2', \theta_2'\right)G\left(x_2', \theta_2'; x_3', \theta_3'\right)\cdots \\ & \times G\left(x_{j-1}', \theta_{j-1}'; x_j', \theta_j'\right)u_0\left(x_j', \theta_j'\right)f\left(x_1',\ldots,x_j'\right)\mathrm{d}x_1'\mathrm{d}\theta_1'\ldots\mathrm{d}x_j'\mathrm{d}\theta_j',\end{aligned} \tag{12}$$

where $f \in L^\infty(B_a \times \cdots \times B_a)$. We take $(x, \theta) \in \Gamma_+$, which corresponds to measuring the specific intensity on $\partial\Omega$ in the outgoing direction.

We will require an estimate on the norm of the operator K_j. We introduce the set $\Gamma_a = B_a \times \mathbb{S}^{d-1}$.

Lemma 2.1. *The operator*

$$K_j : L^\infty\left(B_a \times \cdots \times B_a\right) \longrightarrow L^1\left(\Gamma_+\right)$$

defined by (12) is bounded and

$$\| K_j \| \leqslant \zeta\xi^{j-1},$$

where ξ and ζ are given by

$$\xi = \sigma_0 \sup_{\left(x',\theta'\right)\in\Gamma_a} \int_{\Gamma_a} G\left(x, \theta; x', \theta'\right)\mathrm{d}x\mathrm{d}\theta,$$

$$\zeta = \sigma_0 \int_{\Gamma_a} u_0\mathrm{d}x\mathrm{d}\theta \sup_{\left(x',\theta'\right)\in\Gamma_a} \int_{\Gamma_+} G\left(x, \theta; x', \theta'\right)\mathrm{d}x\mathrm{d}\theta.$$

Proof. The proof is based on repeated application of the Hölder inequality. We begin by observing that

$$\begin{aligned}\| K_j f \|_{L^1(\Gamma_+)} \leqslant \sigma_0^j \int_{\Gamma_+} \int_{\Gamma_a\times\cdots\times\Gamma_a} & G\left(x, \theta; x_1', \theta_1'\right)G\left(x_1', \theta_1'; x_2', \theta_2'\right)\cdots \\ & \times G\left(x_{j-1}', \theta_{j-1}'; x_j', \theta_j'\right)u_0\left(x_j', \theta_j'\right)\left|f\left(x_1', \ldots, x_j'\right)\right|\mathrm{d}x_1'\mathrm{d}\theta_1' \cdots \mathrm{d}x_j'\mathrm{d}\theta_j'\mathrm{d}x\mathrm{d}\theta.\end{aligned} \tag{13}$$

We thus have

$$\| K_j \| \leqslant \sigma_0^j \int_{\Gamma_a} u_0 \mathrm{d}x_j' \mathrm{d}\theta_j' \sup_{(x_j', \theta_j') \in \Gamma_a} \int_{\Gamma_+} \int_{\Gamma_a \times \cdots \times \Gamma_a} G\left(x, \theta; x_1', \theta_1'\right) G\left(x_1', \theta_1'; x_2', \theta_2'\right) \cdots \\ \times G\left(x_{j-1}', \theta_{j-1}'; x_j', \theta_j'\right) \mathrm{d}x_1' \mathrm{d}\theta_1' \cdots \mathrm{d}x_{j-1}' \mathrm{d}\theta_{j-1}' \mathrm{d}x \mathrm{d}\theta. \tag{14}$$

Let us define

$$I_{j-1} = \sigma_0^j \sup_{(x_j', \theta_j') \in \Gamma_a} \int_{\Gamma_+} \int_{\Gamma_a \times \cdots \times \Gamma_a} G\left(x, \theta; x_1', \theta_1'\right) G\left(x_1', \theta_1'; x_2', \theta_2'\right) \cdots \\ \times G\left(x_{j-1}', \theta_{j-1}'; x_j', \theta_j'\right) \mathrm{d}x \mathrm{d}\theta \mathrm{d}x_1' \theta_1' \ldots \mathrm{d}x_{j-1}' \mathrm{d}\theta_{j-1}'. \tag{15}$$

We then have

$$I_{j-1} \leqslant \xi I_{j-2}. \tag{16}$$

Note that I_1 is given by

$$I_1 = \sigma_0^2 \sup_{(x'', \theta'') \in \Gamma_a} \int_{\Gamma_+ \times \Gamma_a} G\left(x, \theta; x', \theta'\right) G\left(x', \theta'; x'', \theta''\right) \mathrm{d}x \mathrm{d}\theta \mathrm{d}x' \mathrm{d}\theta' \\ \leqslant \xi \sigma_0 \sup_{(x', \theta') \in \Gamma_a} \int_{\Gamma_+} G\left(x, \theta; x', \theta'\right) \mathrm{d}x \mathrm{d}\theta. \tag{17}$$

Thus we obtain

$$I_{j-1} \leqslant \xi^{j-1} \sigma_0 \sup_{(x', \theta') \in \Gamma_a} \int_{\Gamma_+} G\left(x, \theta; x', \theta'\right) \mathrm{d}x \mathrm{d}\theta. \tag{18}$$

Putting everything together we obtain

$$\| K_j \| \leqslant \zeta \xi^{j-1}, \tag{19}$$

which completes the proof. □

We now consider the forward problem in $\mathbb{R}^d$ for $d \geqslant 2$. The specific intensity u satisfies

$$\theta \cdot \nabla u(x, \theta) + \sigma(x) u(x, \theta) = \int_{\mathbb{S}^{d-1}} k\left(\theta, \theta'\right) u\left(x, \theta'\right) \mathrm{d}\theta' + q(x, \theta), \quad (x, \theta) \in \mathbb{R}^d \times \mathbb{S}^{d-1}, \tag{20}$$

where q is the source. We measure the specific intensity u on a surface ∂X as shown in figure 1 (right). Correspondingly, we define $\Gamma_+ = \{(x, \theta) \in \partial X \times \mathbb{S}^{d-1} : \theta \cdot n(x) > 0\}$. The solution to (20) is given by the Born series (9), where in the definition of the forward operator K_j, the Green's function G is replaced by the fundamental solution G_0. G_0 obeys (7) with the condition $G_0(x, \theta; x', \theta') \to 0$ as $|x|, |x'| \to \infty$ for all $\theta, \theta' \in \mathbb{S}^{d-1}$. We note that u_0, the solution to (20) with $\eta = 0$, is now given by

$$u_0(x, \theta) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} G_0\left(x, \theta; x', \theta'\right) q\left(x', \theta'\right) \mathrm{d}x' \mathrm{d}\theta'. \tag{21}$$

Remark 2.1. The analysis of the Born series for the case of an infinite medium carries over directly from section 2. In particular, lemma 2.1 and proposition 2.2 both hold.

The fundamental solution G_0 gives an upper bound on the Green's function G as follows.

Lemma 2.2 (**Case–Zweifel** [9]). *Let* $\sigma_0 > 1$. *Then*

$$\int_{\Omega\times\mathbb{S}^{d-1}} G\left(x,\,\theta;\,x',\,\theta'\right)\mathrm{d}x\mathrm{d}\theta \leqslant \int_{\Omega\times\mathbb{S}^{d-1}} G_0\left(x,\,\theta;\,x',\,\theta'\right)\mathrm{d}x\mathrm{d}\theta,\qquad \left(x',\,\theta'\right)\in\Omega\times\mathbb{S}^{d-1}.$$

The proof is a consequence of the fact that $G_0 - G$ satisfies the RTE with nonnegative boundary values $(G_0 - G)|_{\Gamma_-} = G_0$. We see immediately from proposition 2.2 that

$$\xi \leqslant \xi_0,\qquad \zeta \leqslant \zeta_0, \tag{22}$$

where

$$\xi_0 = \sigma_0 \sup_{\left(x',\theta'\right)\in\Gamma_a} \int_{\Gamma_a} G_0\left(x,\,\theta;\,x',\,\theta'\right)\mathrm{d}x\mathrm{d}\theta, \tag{23}$$

$$\zeta_0 = \sigma_0 \int_{\Gamma_a} u_0 \mathrm{d}x\mathrm{d}\theta \sup_{\left(x',\theta'\right)\in\Gamma_a} \int_{\Gamma_+} G_0\left(x,\,\theta;\,x',\,\theta'\right)\mathrm{d}x\mathrm{d}\theta. \tag{24}$$

We thus obtain the following consequence of lemma 2.1, and proposition 2.2.

Proposition 2.1. *The operator*

$$K_j : L^\infty\left(B_a \times \cdots \times B_a\right) \longrightarrow L^1\left(\Gamma_+\right)$$

defined by (*12*) *is bounded and*

$$\| K_j \| \leqslant \zeta_0 \xi_0^{j-1},$$

where ξ_0 *and* ζ_0 *are given by* (*23*) *and* (*24*), *respectively.*

For dimension $d = 3$, we have [14]

$$G_0\left(x,\,\theta;\,x',\,\theta'\right) = \frac{\mathrm{e}^{-\sigma_0|x-x'|}}{\sigma_0\,|x-x'|^2}\delta\left(\theta - \frac{x-x'}{|x-x'|}\right)\delta\left(\theta-\theta'\right)$$
$$+\frac{1}{4\pi\,(2\pi)^3}\int_{\mathbb{R}^3}\mathrm{e}^{\mathrm{i}k\cdot(x-x')}\frac{1}{\left(\sigma_0+\mathrm{i}k\cdot\theta\right)\left(\sigma_0+\mathrm{i}k\cdot\theta'\right)}\left[1-\frac{1}{|k|}\tan^{-1}\left(\frac{|k|}{\sigma_0}\right)\right]^{-1}\mathrm{d}k. \tag{25}$$

Let us evaluate ξ_0 and ζ_0, defined by (23) and (24), for the ball Ω of radius r ($r > a$) and the isotropic delta-function source $q = \delta(x)$ at the origin. By putting $x' = 0$ (ξ_0 is independent of θ'), we obtain

$$\xi_0 = \frac{1-\mathrm{e}^{-\sigma_0 a}}{\sigma_0} + \frac{2\sigma_0 a^2}{\pi}\int_0^\infty \frac{\sin(|k|\,a) - |k|\,a\cos(|k|\,a)}{(|k|\,a)^2}\frac{\left[\tan^{-1}\left(|k|/\sigma_0\right)\right]^2}{|k|-\tan^{-1}(|k|/\sigma_0)}d\,|k|. \tag{26}$$

We obtain ζ_0 as

$$\zeta_0 = \left[\frac{4\pi}{\sigma_0}\left(1-\mathrm{e}^{-\sigma_0 a}\right) + 8a^2\int_0^\infty \frac{\sin(|k|\,a) - |k|\,a\cos(|k|\,a)}{(|k|\,a)^2}\frac{\left[\tan^{-1}\left(|k|/\sigma_0\right)\right]^2}{|k|-\tan^{-1}\left(|k|/\sigma_0\right)}d\,|k|\right]$$
$$\times\left[\frac{1}{\sigma_0^2}\left(1-\mathrm{e}^{-\sigma_0 r}\right) + \frac{2r^2}{\pi}\int_0^\infty \frac{\sin(|k|\,r) - |k|\,r\cos(|k|\,r)}{(|k|\,r)^2}\frac{\left[\tan^{-1}\left(|k|/\sigma_0\right)\right]}{|k|-\tan^{-1}\left(|k|/\sigma_0\right)}d\,|k|\right]. \tag{27}$$

2.1. Convergence of the Born series

As an application of proposition 2.1 we obtain a sufficient condition for the convergence of the Born series (11).

Proposition 2.2. *If the smallness condition* $\| \eta \|_{L^\infty(B_a)} < 1/\xi$ *holds, then the Born series* (*11*) *converges in the* L^1 *norm.*

Proof. We majorize the sum

$$\sum_j \| K_j \eta \otimes \cdots \otimes \eta \|_{L^1(\Gamma_+)} \tag{28}$$

by a geometric series

$$\sum_j \| K_j \eta \otimes \cdots \otimes \eta \|_{L^1(\Gamma_+)} \leqslant \sum_j \| K_j \| \| \eta \|^j_{L^\infty(B_a)} \tag{29}$$

$$\leqslant \zeta \sum_j \xi^{j-1} \| \eta \|^j_{L^\infty(B_a)}, \tag{30}$$

which converges if $\| \eta \|_{L^\infty(B_a)} < 1/\xi$. □

3. Inverse problem

The inverse transport problem is to reconstruct the coefficient η everywhere within Ω from measurements of the scattering data Φ on $\Gamma_+(\Omega)$. In the case of an infinite medium, Φ is assumed to be measured on a surface ∂X containing the support of η, as shown in figure 1. Following [19], we express η as a series in tensor powers of Φ of the form

$$\eta = \mathcal{K}_1 \Phi + \mathcal{K}_2 \Phi \otimes \Phi + \mathcal{K}_3 \Phi \otimes \Phi \otimes \Phi + \cdots, \tag{31}$$

where the inverse operators $\mathcal{K}_j : L^1(\Gamma_+ \times \cdots \times \Gamma_+) \to L^\infty(\Gamma_a)$ are given by

$$\mathcal{K}_1 K_1 = I, \tag{32}$$

$$\mathcal{K}_2 = -\mathcal{K}_1 K_2 \mathcal{K}_1 \otimes \mathcal{K}_1, \tag{33}$$

$$\mathcal{K}_3 = -\left(\mathcal{K}_2 K_1 \otimes K_2 + \mathcal{K}_2 K_2 \otimes K_1 + \mathcal{K}_1 K_3\right) \mathcal{K}_1 \otimes \mathcal{K}_1 \otimes \mathcal{K}_1. \tag{34}$$

For $j \geqslant 2$, we have

$$\mathcal{K}_j = -\left(\sum_{m=1}^{j-1} \mathcal{K}_m \sum_{i_1 + \cdots i_m = j} K_{i_1} \otimes \cdots \otimes K_{i_m} \right) \mathcal{K}_1 \otimes \cdots \otimes \mathcal{K}_1. \tag{35}$$

We will refer to (31) as the inverse Born series.

The operator $\mathcal{K}_1$ is the regularized pseudoinverse of K_1. It is defined as follows. Consider the Tikhonov functional T which is given by

$$T(\eta) = \| K_1 \eta - \phi \|_{L^1(\Gamma_+)} + \alpha F(\eta), \tag{36}$$

where F is a convex penalty function and $\alpha > 0$ is a regularization parameter [23]. The minimizer of T is denoted $\eta^\dagger$ and is referred to as the regularized pseudoinverse solution of

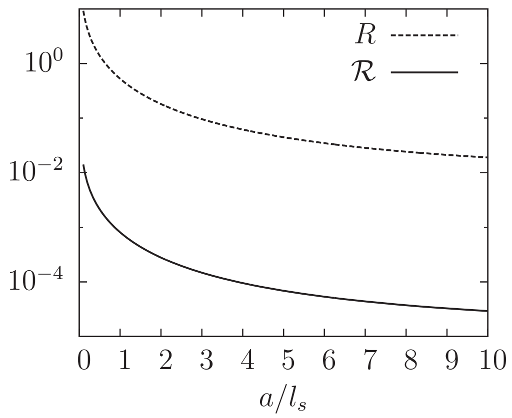

Figure 2. The radii of convergence R and $\mathcal{R}$ for (26) and (27) with $L = 10l_s$.

$K_1\eta = \phi$. The operator $\mathcal{K}_1$ is defined as the map $\mathcal{K}_1 : \phi \mapsto \eta^\dagger$. Here we take $\eta \in X$, where X is a smooth and uniformly convex subspace of $L^\infty(\Omega)$. Since K_1 is bounded, it follows that $\eta^\dagger$ exists and is unique [23, 24].

The inverse Born series for the inverse diffusion problem was analyzed in [19]. It was shown that if the operators K_j obey certain norm estimates then the inverse Born series converges. It is important to note that the limit of the inverse Born series does not, in general, coincide with the coefficient η. We characterize the approximation error as follows.

Theorem 3.1 (Error estimate for the inverse Born series). *Suppose that* $\| \mathcal{K}_1 \| < 1/(\xi_0 + \zeta_0)$ *and* $\| \mathcal{K}_1\Phi \|_{L^\infty(B_a)} < 1/(\xi_0 + \zeta_0)$. *Let* $M = \max(\| \eta \|_{L^\infty(B_a)}, \| \mathcal{K}_1 K_1 \eta \|_{L^\infty(B_a)})$ *and assume that* $M < 1/(\xi_0 + \zeta_0)$. *Then the inverse Born series (31) converges and the following error estimate holds:*

$$\left\| \eta - \sum_{j=1}^{\infty} \mathcal{K}_j \Phi \otimes \cdots \otimes \Phi \right\|_{L^\infty(B_a)} \leqslant C \left\| \left(I - \mathcal{K}_1 K_1\right)\eta \right\|_{L^\infty(B_a)},$$

where $C = C(\xi_0, \zeta_0, \| \mathcal{K}_1 \|, M)$.

The proof of theorem 3.1 makes use of the norm estimates (2.1) and follows the same approach as the proof of theorem 3.1 in [19].

Remark 3.1. We emphasize that $\mathcal{K}_1$ is the regularized pseudoinverse of K_1. As a consequence, the inverse Born series does not converge to η. However, if η is known *a priori* to belong to the subspace on which it is possible to invert K_1, then the limit of the series coincides precisely with η. Further discussion of this point is provided in [19].

3.1. Convergence

We define the radii of convergence

$$R = \frac{1}{\xi}, \quad \mathcal{R} = \frac{1}{\xi + \zeta}. \tag{37}$$

We will refer to R as the radius of convergence of the Born series and $\mathcal{R}$ as the radius of convergence of the inverse Born series.

Figure 2 illustrates the convergence of the Born and inverse Born series in three-dimensional space. We plot R and $\mathcal{R}$ for (26) and (27) with $\sigma_0 = 1.01$ and $L = 10l_s$, where l_s is the scattering length. Note that the unit of length is l_s in the RTE (1). We see that R and $\mathcal{R}$ decay rapidly as functions of the radius a of the ball B_a.

4. Angularly-averaged measurements

As noted in the introduction, angularly-resolved measurements of the specific intensity are difficult to obtain in practice. Instead, angularly-averaged intensity measurements of the form

$$I(x) = \int_{\theta\cdot n>0} \theta \cdot nA(\theta)u(x, \theta)\mathrm{d}\theta \tag{38}$$

are often considered. Note that $A = 1$ for fully-averaged measurements. In the case of an aperture which selects photons traveling in the outward normal direction, we have $A = \delta(\theta - n)$.

It is straightforward to modify the Born series to account for the effect of angular averaging. equation (11) becomes

$$\Psi = \mathscr{K}_1\eta + \mathscr{K}_2\eta \otimes \eta + \mathscr{K}_3\eta \otimes \eta \otimes \eta + \cdots. \tag{39}$$

Here the angularly average scattering data Ψ and the operators $\mathscr{K}_j$ are defined by

$$\Psi(x) = \int_{\theta\cdot n>0} \theta \cdot nA(\theta)\Phi(x, \theta)\mathrm{d}\theta, \tag{40}$$

$$\left(\mathscr{K}_j f\right)(x) = \int_{\theta\cdot n>0} \theta \cdot nA(\theta)\left(K_j f\right)(x, \theta)\mathrm{d}\theta, \quad x \in \partial\Omega, \tag{41}$$

where $f \in L^\infty(B_a \times \cdots \times B_a)$. The following result is an immediate consequence of the proof of lemma 2.1 and proposition 2.1.

Proposition 4.1. *The operator*

$$\mathscr{K}_j : L^\infty\left(B_a \times \cdots \times B_a\right) \longrightarrow L^1(\partial\Omega)$$

defined by (41) is bounded and

$$\| \mathscr{K}_j \|_\infty \leqslant \zeta_A \xi_0^{j-1},$$

where ξ_0 is given by (23) and

$$\zeta_A = \sigma_0 \int_{\Gamma_a} u_0 \mathrm{d}x\mathrm{d}\theta \sup_{(x',\theta')\in\Gamma_a} \int_{\Gamma_+} |\theta \cdot n|\, A(\theta)G\left(x, \theta; x', \theta'\right)\mathrm{d}x\mathrm{d}\theta.$$

The inverse problem with angularly-averaged measurements is to recover the coefficient η from measurements of Ψ on $\partial\Omega$. The inverse of the Born series (39) for the angularly-averaged scattering data Ψ has the form (31), where the scattering data Φ is replaced by Ψ and the $\mathcal{K}_j$ are constructed according to (35), with $\mathscr{K}_j$ taking the place of K_j. It follows from proposition 4.1 that theorem 3.1 holds.

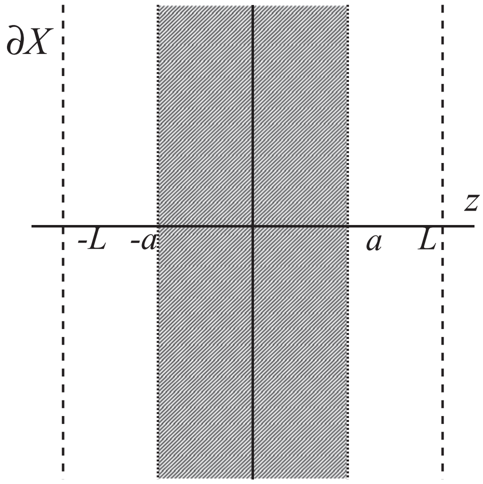

Figure 3. The slab problem in section 5.

5. Slab problem

In this section we consider the slab problem, for which we will implement and test the inverse Born series reconstruction. The problem is defined in section 5.1 where we also introduce some useful facts about so-called singular eigenfunctions of the RTE. The corresponding Born and inverse Born series are then constructed in section 5.2. Our numerical results are presented in section 5.4.

5.1. Setup

Let us consider a homogeneous slab-shaped medium with attenuation σ_1 embedded in a homogeneous infinite medium with attenuation σ_0, where $\sigma_1 > \sigma_0$. See figure 3. We suppose the embedded medium occupies the strip $-a \leqslant z \leqslant a$. We consider an isotropic plane-parallel source on the planes ∂X of the form

$$q(z, \theta) = \delta(z + L) + \delta(z - L), \tag{42}$$

where $L > a$. The specific intensity $u(z, \mu)$ is assumed to be measured on the planes at $z = -L, L$. We also assume that the medium is isotropically scattering, with $k = 1/(4\pi)$. In this setting [7, 13], the RTE (20) becomes

$$\mu\frac{\partial u}{\partial z} + \sigma(z)u = \frac{1}{2}\int_{-1}^{1} u\left(z, \mu'\right)\mathrm{d}\mu' + q(z, \theta), \quad (z, \mu) \in (-\infty, \infty) \times [-1, 1], \tag{43}$$

where $\mu = \cos\theta$, with θ the usual polar angle in spherical coordinates and $u \to 0$ as $|z| \to \infty$. Here the coefficient σ is given by

$$\sigma(z) = \sigma_0(1 + \eta(z)), \quad \eta(z) = \begin{cases} \eta_1 & z \in [-a, a], \\ 0 & z \notin [-a, a], \end{cases} \quad \eta_1 = \frac{\sigma_1}{\sigma_0} - 1. \tag{44}$$

We note that (43) is invariant under the transformation $(z, \mu) \to (-z, -\mu)$ and hence $u(-z, -\mu) = u(z, \mu)$. Therefore with this mirror symmetry, we can restrict our attention to the half-space ($z > 0$). The RTE (43) thus becomes

$$\mu\frac{\partial u}{\partial z}+\sigma(z)u=\frac{1}{2}\int_{-1}^{1}u\left(z,\,\mu'\right)\mathrm{d}\mu'+\delta(z-L),\qquad 0<z<\infty,\tag{45}$$

$$u(0,\,-\mu)=u(0,\,\mu),\tag{46}$$

$$u(z,\,\mu)\to 0,\qquad z\to\infty.\tag{47}$$

The Green's function $G(z,\,\mu;\,z',\,\mu')$ corresponding to (45) obeys

$$\mu\frac{\partial G}{\partial z}+\sigma_0 G=\frac{1}{2}\int_{-1}^{1}G\left(z,\,\mu'';\,z',\,\mu'\right)\mathrm{d}\mu''+\delta(z-z')\delta\left(\mu-\mu'\right).\tag{48}$$

It is known that the solution to the RTE (45) can be expressed as a superposition of eigenmodes $\phi(\tau,\,\mu)$, which are known as singular eigenfunctions [8, 9]. Here τ takes either discrete values $\pm\nu_0$ ($\nu_0>1$) or lies in the continuous spectrum between -1 and 1. The construction of the singular eigenfunctions is outlined in the appendix. We can now express u_0 as

$$u_0(z,\,\mu)=\bar{u}_0(\nu_0,\,z)\phi(\nu_0,\,\mu)+\bar{u}_0(-\nu_0,\,z)\phi(-\nu_0,\,\mu)+\int_{-1}^{1}\bar{u}_0(\nu,\,z)\phi(\nu,\,\mu)\mathrm{d}\nu\tag{49}$$

$$=\sum_{\tau\in\Gamma_\tau}\bar{u}_0(\tau,\,z)\phi(\tau,\,\mu),\tag{50}$$

where $\bar{u}_0$ are Fourier coefficients and

$$\Gamma_\tau=\left\{\tau\in\mathbb{R}:\ \tau=\pm\,\nu_0,\ \ \tau\in(-1,\,1)\right\}.\tag{51}$$

The expansion (49) is referred to as the Case transform [18].

5.2. The Born and inverse Born series

The Born series corresponding to the RTE (43) is of the form

$$\begin{aligned}\Phi(\mu)&:=u_0(L,\,\mu)-u(L,\,\mu)\\&=\left(K_1\eta\right)(\mu)+\left(K_2\eta\otimes\eta\right)(\mu)+\cdots,\end{aligned}\tag{52}$$

where

$$u_0(z,\,\mu)=\int_{-1}^{1}G\left(z,\,\mu;\,L,\,\mu'\right)\mathrm{d}\mu'\tag{53}$$

and Φ is the data function, whose dependence on the parameter L is not indicated. The operator K_j is defined by

$$\begin{aligned}\left(K_j f\right)(\mu)=(-1)^{j+1}\sigma_0^j\int_{\Gamma_a\times\cdots\times\Gamma_a}\int_{-1}^{1}G\left(L,\,\mu;\,z_1',\,\mu_1'\right)G\left(z_1',\,\mu_1';\,z_2',\,\mu_2'\right)\cdots\\\times\,G\left(z_{j-1}',\,\mu_{j-1}';\,z_j',\,\mu_j'\right)G\left(z_j',\,\mu_j';\,L,\,\mu'\right)f\left(z_1',\,\ldots,\,z_j'\right)\mathrm{d}z_1'\mathrm{d}\mu_1'\cdots\mathrm{d}z_j'\mathrm{d}\mu_j'\mathrm{d}\mu',\end{aligned}\tag{54}$$

where $f\in L^\infty([0,\,a]\times\cdots\times[0,\,a])$ and $\Gamma_a=[0,\,a]\times[-1,\,1]$.

It will prove convenient, for both mathematical and computational reasons, to work with the Born series expressed in terms of singular eigenfunctions. To proceed, we expand Φ into singular eigenfunctions ϕ:

$$\Phi(\mu)=\sum_{\tau\in\Gamma_\tau}\bar{\Phi}(\tau)\phi(\tau,\,\mu),\tag{55}$$

where $\bar{\Phi}(\tau)$ are expansion coefficients. We can calculate $\bar{\Phi}(\tau)$ by using the orthogonality relations (A.5)–(A.7). Likewise, we introduce the corresponding transformed forward operators by $\bar{K}_j : L^\infty([0, a] \times \cdots \times [0, a]) \longrightarrow L^1(\Gamma_\tau)$. The $\bar{K}_j$ are defined by the relations

$$\begin{aligned}\left(K_j \eta \otimes \cdots \otimes \eta\right)(\mu) = &\sum_\tau \phi(\tau, \mu) \\ &\times \int_0^a \cdots \int_0^a \bar{K}_j\left(\tau, z_1', \ldots, z_j'\right) \eta\left(z_1'\right) \cdots \eta\left(z_j'\right) \mathrm{d}z_1' \cdots \mathrm{d}z_j'. \end{aligned} \tag{56}$$

Using the above, we see that the transformed Born series is of the form

$$\bar{\Phi} = \bar{K}_1 \eta + \bar{K}_2 \eta \otimes \eta + \bar{K}_3 \eta \otimes \eta \otimes \eta + \cdots. \tag{57}$$

The inverse problem now consists of recovering the coefficient η from the data $\bar{\Phi}$. The inverse of the Born series (57) has the form (31), where the scattering data Φ is replaced by $\bar{\Phi}$ and the $\mathcal{K}_j$ are constructed according to (35). That is,

$$\begin{aligned}\eta &= \eta^{(1)} + \eta^{(2)} + \eta^{(3)} + \cdots \\ &= \bar{\mathcal{K}}_1 \bar{\Phi} + \bar{\mathcal{K}}_2 \bar{\Phi} \otimes \bar{\Phi} + \bar{\mathcal{K}}_3 \bar{\Phi} \otimes \bar{\Phi} \otimes \bar{\Phi} + \cdots, \end{aligned} \tag{58}$$

where the inverse operators $\bar{\mathcal{K}}_j : L^1(\Gamma_\tau \times \cdots \times \Gamma_\tau) \longrightarrow L^\infty([0, a])$ are given by (35) with $\bar{\mathcal{K}}_j$ taking the place of K_j.

5.3. Reconstruction algorithm

To proceed with numerical reconstructions, we discretize the spatial variable z and the continuous spectrum ν as

$$z_j = (j - 1)\Delta_z, \quad \Delta_z = \frac{L}{N_z - 1}, \quad \nu_i = \mathrm{i}\Delta_\nu, \quad \Delta_\nu = \frac{1}{N_\nu + 1}, \tag{59}$$

where $j = 1, 2, \ldots, N_z$ and $i = 1, 2, \ldots, N_\nu$. We define $N_\tau = 2N_\nu + 2$ (the number of positive and negative continuous eigenvalues, and positive and negative discrete eigenvalues).

The reconstructed coefficient η, to jth order in the inverse Born series, is calculated from $\bar{\Phi}$ using the following forward function K and inverse function $\mathcal{K}$. We first calculate the function K:

$$\varphi = K\left(j, \eta_1, \ldots, \eta_j\right), \tag{60}$$

where j is an integer and $\eta_1, \ldots, \eta_j$ are vectors of size N_z. The function returns a vector φ of size N_τ. Each element $\{\varphi\}_\tau$ is given by

$$\{\varphi\}_\tau = \left\{\bar{K}_j \eta_1 \otimes \cdots \otimes \eta_j\right\}_\tau = \bar{\Phi}^{(j)}(\tau), \tag{61}$$

where $\bar{\Phi}^{(j)}(\tau)$ is computed in (A.24).

Next we construct the inverse function $\mathcal{K}$. The function has an integer j and j vectors of size N_τ as its arguments and returns the vector η of size N_z:

$$\eta = \mathcal{K}\left(j, \bar{\Phi}_1, \ldots, \bar{\Phi}_j\right). \tag{62}$$

The calculation is performed according to the following steps.

Step 1. For each $\bar{\Phi}_i$ $(1 \leqslant i \leqslant j)$, we compute the linear reconstruction

$$\eta_i = \bar{\mathcal{K}}_1 \bar{\Phi}_i, \tag{63}$$

where η_i is a vector of size N_z and the matrix $\bar{\mathcal{K}}_1$ is computed by singular value decomposition as $\bar{\mathcal{K}}_1 = \bar{K}_1^+$. If $j = 1$, the function returns $\eta = \eta_1$. In what follows, we assume $j > 1$.

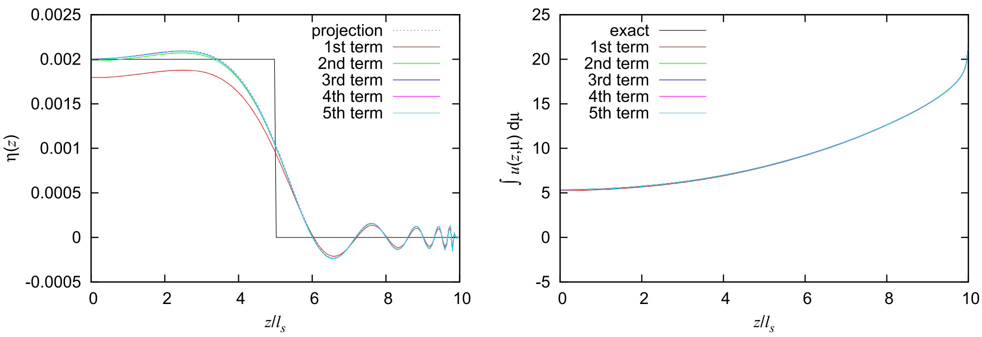

Figure 4. (Left) reconstructions of η to fifth order in the inverse Born series are shown. (Right) the energy density $\int_{-1}^{1} u(z, \mu)\mathrm{d}\mu$, computed to fifth order in the Born series, is compared to the energy density obtained from the exact solution to the forward problem. In both panels the contrast $\Delta = 1.2$.

Step 2. We form the compositions $[i_1, ..., i_m]$ such that $i_1 + \cdots + i_m = j$. For each m ($1 \leqslant m \leqslant j - 1$) and each composition $(i_1, ..., i_m)$, by recursively using the function $\mathcal{K}$ in (62), we compute

$$\eta_{\mathrm{tmp}} = \mathcal{K}\Big(m, K\left(i_1, \eta_1, ..., \eta_{i_1}\right), ..., K\left(i_m, \eta_{j-i_m+1}, ..., \eta_j\right)\Big). \tag{64}$$

Let Σ_2 denote the sum of η_{tmp} for all $\begin{pmatrix} j-1 \\ m-1 \end{pmatrix}$ compositions.

Step 3. Step 2 is repeated for all m ($1 \leqslant m \leqslant j - 1$). Let Σ_1 denote the sum of the results from step 2:

$$\Sigma_1 = \sum_{m=1}^{j-1} \Sigma_2. \tag{65}$$

Finally, the function $\mathcal{K}$ returns $\eta = \Sigma_1$. In particular, if we take the first N terms in the inverse Born series, the jth reconstructed term $\eta^{(j)}$ for the scattering data $\bar{\Phi}$ is calculated as

$$\eta^{(j)} = \mathcal{K}\big(j, \bar{\Phi}, ..., \bar{\Phi}\big), \tag{66}$$

where η is approximated as $\eta^{(1)} + \eta^{(2)} + \cdots + \eta^{(N)}$.

5.4. Numerical results

We now discuss numerical tests of the reconstruction scheme developed above. To proceed, we require the forward data $\bar{\Phi}$, which can be obtained by solving the RTE using singular eigenfunctions in the slab geometry. This calculation has been reported in [7, 13].

The parameters in the simulated reconstructions were chosen as follows. Let $\mu_s = 1/l_s$ denote the scattering coefficient. Together with the absorption coefficients μ_{a0} and μ_{a1}, we have $\sigma_0 = 1 + \mu_{a0}/\mu_s$ and $\sigma_1 = 1 + \mu_{a1}/\mu_s$. The contrast in absorption is defined by

$$\Delta = \frac{\mu_{a1}}{\mu_{a0}} = 1 + \frac{\eta_1 \sigma_0}{\sigma_0 - 1}. \tag{67}$$

In biological tissue $1 < \sigma_0 \ll 2$ [1]. We put $\sigma_0 = 1.01$ and vary η_1 as follows: $\eta_1 = 0.002$, 0.007, 0.013, 0.019, 0.07, 0.19, which corresponds to $\sigma_1 = 1.012$, 1.017, 1.023, 1.029, 1.08, 1.2. The contrast is thus varied over the range $\Delta = 1.2$, 1.7, 2.3, 2.9, 8.0, 20.

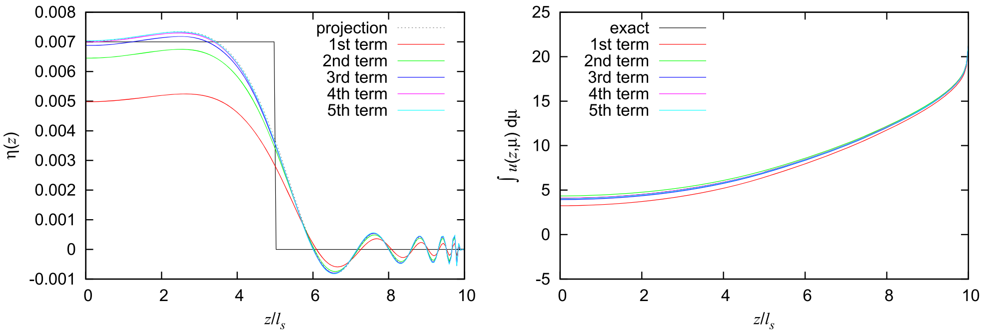

Figure 5. As in figure 4, except that the contrast $\Delta = 1.7$.

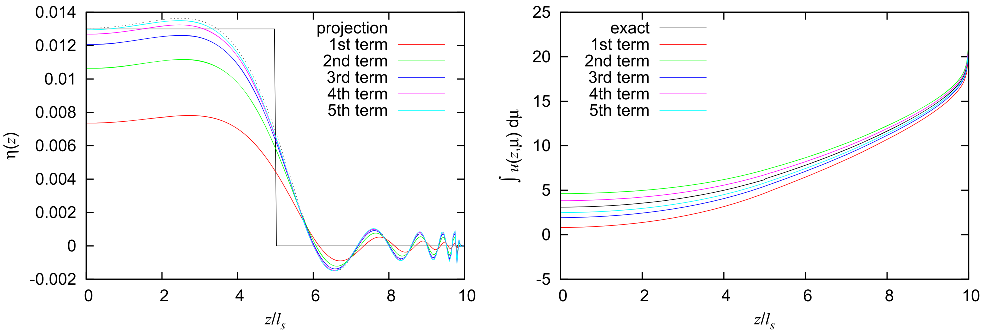

Figure 6. As in figure 4, except that the contrast $\Delta = 2.3$.

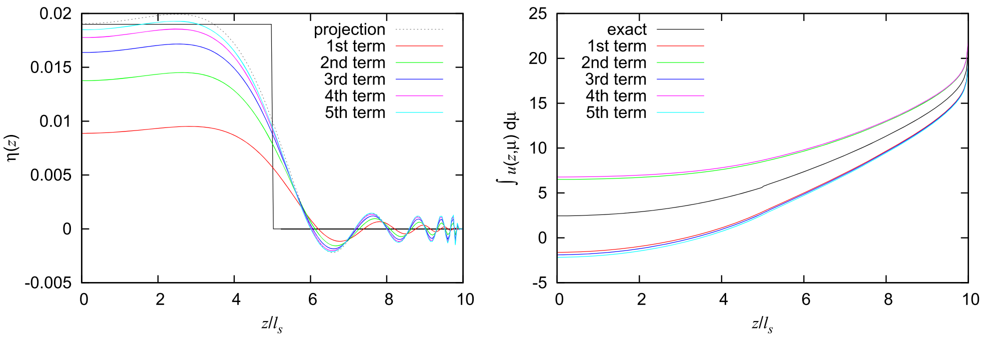

Figure 7. As in figure 4, except that the contrast $\Delta = 2.9$.

In the simulations shown below, the source and detector are placed at $L = 10l_s$. In addition we take $a = 5l_s$, where a is the radius of the ball B_a. The Moore–Penrose inverse of K_1, computed by singular value decomposition, is used to compute $\bar{\mathcal{K}}_1 = \bar{K}_1^+$, consistent with (36). The number of spatial discretization points is taken to be $N_z = 200$ and the number of discretization points of the continuous spectrum is $N_\nu = 51$. We found that increasing the number of discretization points did not significantly affect the reconstructions.

Numerical reconstructions of the coefficient η, at various levels of contrast, are shown in figures 4 through 9. In all cases, the inverse Born series is calculated to fifth order. The

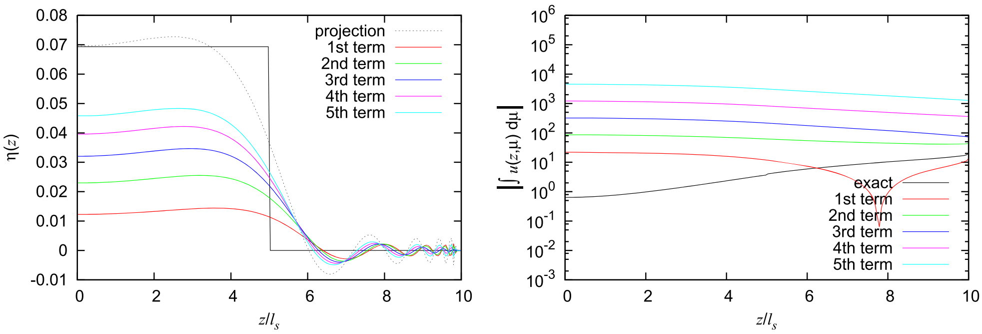

Figure 8. As in figure 4, except that the contrast $\Delta = 8.0$.

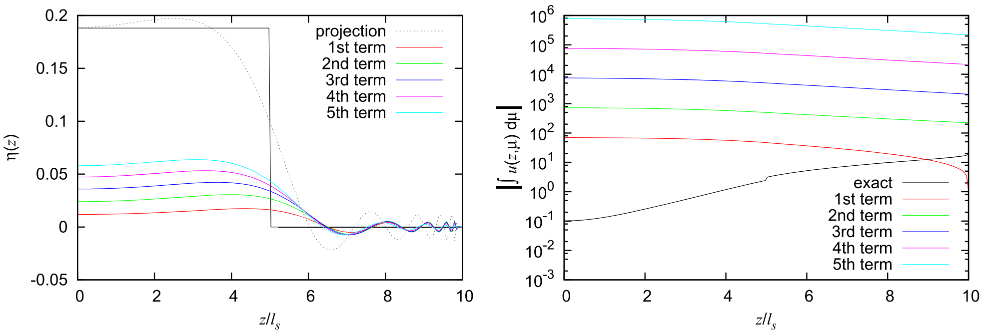

Figure 9. As in figure 4, except that the contrast $\Delta = 20$.

projection $\eta_{\text{proj}} = \mathcal{K}_1^{+} K_1 \eta$ is also plotted in each figure. In some sense, the projection is the best approximation to η that can be expected. Also shown, in each figure, is a comparison of the exact solution to the forward problem with the Born series computed to fifth order. In figures 4 and 5 we show low contrast reconstructions with $\Delta = 1.2, 1.7$. It can be seen that the series appears to converge rapidly. Next, in figures 6 and 7 we present reconstructions at intermediate contrast with $\Delta = 2.3, 2.9$. These values of Δ are typical in optical tomography. In this case, the series converges more slowly. Finally, in figures 8 and 9 we show reconstructions at high contrast with $\Delta = 8, 20$. Although there is some improvement at fifth order compared to the linear reconstruction, it is evident that the both the the inverse Born series and the Born series have not converged.

6. Discussion

In conclusion, we have investigated the inverse Born series for the inverse transport problem. We have analyzed the approximation error of the series and have conducted confirmatory numerical simulations for a slab-shaped medium. Exact solutions to the forward problem were used as scattering data and reconstructions were computed to fifth order in the inverse series. We found that the series appears to converge quite rapidly for low contrast objects and that as the contrast is increased, the higher order terms systematically improve the reconstructions until, at sufficiently large contrast, the series diverges. We expect to test these conclusions with data from experiments in the near future. It will then be important to compare the inverse Born series with optimization-based reconstruction methods. We note that for the Calderon problem, reconstructions using the inverse Born series were compared to those from the Gauss–Newton method [3]. It was found that the quality of reconstructions was comparable in both cases. However, since the inverse Born series does not make use of a PDE-based forward solver, its computational cost is less than the Gauss–Newton method. Finally, we note that the interplay between the extent of angular averaging, regularization and resolution will be important to explore in future work, as will be a more extensive analysis of computational complexity.

Acknowledgments

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Appendix A. Singular eigenfunctions

Here we express the forward operators K_j and inverse operators $\mathcal{K}_j$ defined in section 5 using singular eigenfunctions and obtain the transformed forward and inverse operators $\bar{K}_j$ and $\bar{\mathcal{K}}_j$.

Since K_j and $\mathcal{K}_j$ are given by the Green's function for the homogeneous medium with attenuation σ_0, we first consider G in (48). The singular eigenfunctions ϕ are defined by [8, 9]

$$\phi\left(\pm\nu_0,\, \mu\right) = \pm\frac{\nu_0}{2\sigma_0}\frac{1}{\pm\nu_0 - \mu}, \quad \phi(\nu,\, \mu) = \frac{\nu}{2\sigma_0}\mathcal{P}\frac{1}{\nu - \mu} + \lambda(\nu)\delta(\nu - \mu), \tag{A.1}$$

where $\nu \in (-1,\, 1)$, $\mathcal{P}$ denotes the Cauchy principal value and

$$\lambda(\nu) = 1 - \frac{\nu}{\sigma_0}\tanh^{-1}(\nu). \tag{A.2}$$

The discrete eigenvalue $\nu_0 > 1$ is the positive solution to the transcendental equation

$$\nu_0 = \sigma_0 \Big/ \tanh^{-1}\left(\frac{1}{\nu_0}\right). \tag{A.3}$$

The singular eigenfunctions satisfy the following normalization and orthogonality relations:

$$\int_{-1}^{1} \phi(\tau,\, \mu)\mathrm{d}\mu = 1, \tag{A.4}$$

$$\int_{-1}^{1} \mu\phi(\tau,\, \mu)\phi\left(\tau',\, \mu\right)\mathrm{d}\mu = 0, \quad \tau \neq \tau', \tag{A.5}$$

$$\int_{-1}^{1} \mu\phi(\pm\nu_0,\, \mu)\phi(\pm\nu_0,\, \mu)\mathrm{d}\mu = \pm\nu_0\mathcal{N}_0, \tag{A.6}$$

$$\int_{-1}^{1} \mu\phi(\nu,\, \mu)\phi\left(\nu',\, \mu\right)\mathrm{d}\mu = \nu\mathcal{N}(\nu)\delta\left(\nu - \nu'\right), \tag{A.7}$$

where

$$\mathcal{N}_0 = \frac{\nu_0^2}{2\sigma_0}\left(\frac{1}{\sigma_0\left(\nu_0^2 - 1\right)} - \frac{1}{\nu_0^2}\right), \quad \mathcal{N}(\nu) = \left(1 - \frac{\nu}{\sigma_0}\tanh^{-1}\nu\right)^2 + \frac{\pi^2\nu^2}{4\sigma_0^2}. \tag{A.8}$$

Using the expansion coefficients $g_\tau(z,\, z')$, which will be determined below, we can write the Green's function G in (48) as

$$
\begin{aligned}
G\left(z, \mu; z', \mu'\right) &= g_{\nu_0}(z, z')\phi(\nu_0, \mu)\phi\left(\nu_0, \mu'\right) + g_{-\nu_0}(z, z')\phi(-\nu_0, \mu)\phi\left(-\nu_0, \mu'\right) \\
&\quad + \int_{-1}^{1} g_\nu(z, z')\phi(\nu, \mu)\phi\left(\nu, \mu'\right)\mathrm{d}\nu \\
&= \sum_\tau g_\tau(z, z')\phi(\tau, \mu)\phi\left(\tau, \mu'\right),
\end{aligned}
\tag{A.9}
$$

where $\tau \in \Gamma_\tau$ (recall (51)). To find g_τ, we note that G can be written using the one-dimensional free-space Green's function G_{01}, which satisfies (48) for $-\infty < r < \infty$:

$$
\begin{aligned}
&G\left(z, \mu; z', \mu'\right) \\
&= G_{01}\left(z, \mu; z', \mu'\right) + a_+\phi(\nu_0, \mu)\mathrm{e}^{-\sigma_0 z/\nu_0} + \int_0^1 A(\nu)\phi(\nu, \mu)\mathrm{e}^{-\sigma_0 z/\nu}\mathrm{d}\nu.
\end{aligned}
\tag{A.10}
$$

The coefficients a_+ and $A(\nu)$ are determined by the symmetry condition $G(0, \mu; z', \mu') = G(0, -\mu; z', \mu')$, and are obtained as

$$
\begin{aligned}
a_+ &= \frac{1}{\nu_0 \mathcal{N}_0}\mathrm{e}^{-\sigma_0 z'/\nu_0}\phi\left(\nu_0, \mu'\right), \\
A(\nu) &= \frac{1}{\nu \mathcal{N}(\nu)}\mathrm{e}^{-\sigma_0 z'/\nu}\phi\left(\nu, \mu'\right)\mathrm{d}\nu.
\end{aligned}
\tag{A.11}
$$

The Green's function G_{01} is given by [9]

$$
G_{01}\left(z, \mu; z', \mu'\right) = \begin{cases}
\dfrac{1}{\nu_0 \mathcal{N}_0}\mathrm{e}^{-\sigma_0(z-z')/\nu_0}\phi(\nu_0, \mu)\phi\left(\nu_0, \mu'\right) & \\
\quad + \displaystyle\int_0^1 \frac{1}{\nu \mathcal{N}(\nu)}\mathrm{e}^{-\sigma_0(z-z')/\nu}\phi(\nu, \mu)\phi\left(\nu, \mu'\right)\mathrm{d}\nu & z > z', \\
\dfrac{1}{\nu_0 \mathcal{N}_0}\mathrm{e}^{-\sigma_0|z-z'|/\nu_0}\phi(-\nu_0, \mu)\phi\left(-\nu_0, \mu'\right) & \\
\quad + \displaystyle\int_0^1 \frac{1}{\nu \mathcal{N}(\nu)}\mathrm{e}^{-\sigma_0|z-z'|/\nu}\phi(-\nu, \mu)\phi\left(-\nu, \mu'\right)\mathrm{d}\nu & z < z'.
\end{cases}
\tag{A.12}
$$

Therefore the functions $g_\tau(z, z')$ $(\nu > 0)$ are obtained as

$$
g_{\nu_0}(z, z') = \frac{1}{\nu_0 \mathcal{N}_0}\left[\mathrm{e}^{-\sigma_0(z+z')/\nu_0} + \Theta(z - z')\mathrm{e}^{-\sigma_0(z-z')/\nu_0}\right],
\tag{A.13}
$$

$$
g_{-\nu_0}(z, z') = \Theta(z' - z)\frac{1}{\nu_0 \mathcal{N}_0}\mathrm{e}^{-\sigma_0|z-z'|/\nu_0},
\tag{A.14}
$$

$$
g_\nu(z, z') = \frac{1}{\nu \mathcal{N}(\nu)}\left[\mathrm{e}^{-\sigma_0(z+z')/\nu} + \Theta(z - z')\mathrm{e}^{-\sigma_0(z-z')/\nu}\right],
\tag{A.15}
$$

$$
g_{-\nu}(z, z') = \Theta(z' - z)\frac{1}{\nu \mathcal{N}(\nu)}\mathrm{e}^{-\sigma_0|z-z'|/\nu}.
\tag{A.16}
$$

Here $\Theta(\cdot)$ is the step function.

Since the specific intensity u is measured on the plane $z = L$, (52) reads

$$\Phi(L, \mu) = \sigma_0 \int_0^L \int_{-1}^1 G\left(L, \mu; z', \mu'\right) u_0\left(z', \mu'\right) \eta(z') \mathrm{d}z' \mathrm{d}\mu' + \cdots. \tag{A.17}$$

We write the Born series as $\Phi = \Phi^{(0)} + \Phi^{(1)} + \cdots$ and introduce $\bar{\Phi}^{(j)}$ as

$$\Phi^{(j)}(L, \mu) = \sum_\tau \bar{\Phi}^{(j)}(\tau) \phi(\tau, \mu). \tag{A.18}$$

Thus, using (A.1) and (A.9), the jth term in the series is given by

$$\bar{\Phi}^{(j)}(\tau) = \int_0^a \dots \int_0^a \bar{K}_j\left(\tau, z_1', \dots, z_j'\right) \eta\left(z_1'\right) \cdots \eta\left(z_j'\right) \mathrm{d}z_1' \cdots \mathrm{d}z_j'. \tag{A.19}$$

Here $\bar{K}_j : L^\infty([0, a] \times \cdots \times [0, a]) \longrightarrow L^1(\Gamma_\tau)$ are given by

$$\begin{aligned} &\bar{K}_j\left(\tau, z_1', \dots, z_j'\right) = (-1)^{j+1} \sigma_0^j \\ &\times \sum_{\tau_1, \dots, \tau_j} g_\tau\left(L, z_1'\right) h(\tau, \tau_1) g_{\tau_1}\left(z_1', z_2'\right) \cdots h(\tau_{j-1}, \tau_j) g_{\tau_j}\left(z_j', L\right), \end{aligned} \tag{A.20}$$

where we have defined

$$\begin{aligned} h\left(\tau, \tau'\right) &= \int_{-1}^1 \phi(\tau, \mu) \phi\left(\tau', \mu\right) \mathrm{d}\mu \\ &= \begin{cases} \mathcal{N}_0 \delta_{\tau, \tau'} + \dfrac{1}{2\sigma_0}, & \tau = \pm\nu_0, \\ \mathcal{N}(\nu) \delta\left(\nu - \tau'\right) + \dfrac{1}{2\sigma_0}, & \tau = \nu \quad (-1 < \nu < 1). \end{cases} \end{aligned} \tag{A.21}$$

Thus we obtain the Born series for $\bar{\Phi}$ in (57).

Each term $\bar{\Phi}^{(j)}$ in the Born series can be recursively calculated as follows. Let us introduce $\mathcal{L}_i$ $(2 \leqslant i \leqslant j)$ as

$$\mathcal{L}_i\left(\tau', z'\right) = -\sigma_0 \int_0^a \sum_{\tau''} \mathcal{L}_{i-1}\left(\tau'', z''\right) h\left(\tau'', \tau'\right) g_{\tau'}(z'', z') \eta_i(z'') \mathrm{d}z'', \tag{A.22}$$

where the initial term is given by

$$\mathcal{L}_1\left(\tau', z'\right) = \sigma_0 \int_0^a g_\tau(L, z'') h\left(\tau, \tau'\right) g_{\tau'}(z'', z') \eta_1(z'') \mathrm{d}z''. \tag{A.23}$$

From (A.19), we see that

$$\bar{\Phi}^{(j)}(\tau) = \sum_{\tau'} \mathcal{L}_j\left(\tau', L\right). \tag{A.24}$$

The inverse Born series (58) is now written as

$$\eta(z) = \sum_\tau \bar{\mathcal{K}}_1(z, \tau) \bar{\Phi}(\tau) + \sum_{\tau_1, \tau_2} \bar{\mathcal{K}}_2(z, \tau_1, \tau_2) \bar{\Phi}(\tau_1) \otimes \bar{\Phi}(\tau_2) + \cdots, \tag{A.25}$$

where $\bar{\mathcal{K}}_j$ are the corresponding inverse operators.

References

[1] Arridge S R 1999 Optical tomography in medical imaging *Inverse Problems* **15** R41
[2] Arridge S R and Schotland J C 2009 Optical tomography: forward and inverse problems *Inverse Problems* **25** 123010
[3] Arridge S R, Moskow S and Schotland J C 2012 Inverse Born series for the Calderon problem *Inverse Problems* **28** 035003
[4] Bal G 2009 Inverse transport theory and applications *Inverse Problems* **25** 053001
[5] Bal G and Jollivet A 2008 Stability estimates in stationary inverse transport *Inverse Probl. Imaging* **2** 427
[6] Bal G, Langmore I and Monard F 2008 Inverse transport with isotropic sources and angularly averaged measurements *Inverse Probl. Imaging* **1** 23
[7] Bond G R and Siewert C E 1969 The effect of linearly anisotropic neutron scattering on disadvantage factor calculations *Nucl. Sci. Eng.* **35** 277
[8] Case K M 1960 Elementary solutions of the transport equation and their applications *Ann. Phys.* **9** 1
[9] Case K M and Zweifel P F 1967 *Linear Transport Theory* (Reading, MA: Addison-Wesley)
[10] Choulli M and Stefanov P 1996 Reconstruction of the coefficients of the stationary transport equation from boundary measurements *Inverse Problems* **12** L19
[11] Choulli M and Stefanov P 1996 Inverse scattering and inverse boundary value problems for the linear Boltzmann equation *Commun. PDE* **21** 76385
[12] Choulli M and Stefanov P 1999 An inverse boundary value problem for the stationary transport equation *Osaka J. Math.* **36** 87
[13] Eccleston G W and McCormick N J 1970 One-speed transport disadvantage factor calculations for general anisotropic scattering *J. Nucl. Eng.* **24** 23
[14] Ishimaru A 1977 *Wave Propagation and Scattering in Random Media* vol 1 (Piscataway, NJ: IEEE)
[15] Kilgore K, Moskow S and Schotland J C 2009 Inverse Born series for diffuse waves *Contemp. Math.* **494** 113–23
[16] Langmore I 2008 The stationary transport equation with angularly averaged measurements *Inverse Problems* **24** 015924
[17] Langmore I and McDowall S 2008 Optical tomography for variable refractive index with angularly averaged measurements *Commun. PDE* **33** 2180–207
[18] Larsen E W, Sancaktar S and Zweifel P F 1975 Extension of the Case formulas to L_p. Application to half and full space problems *J. Math. Phys.* **16** 1117–21
[19] Moskow S and Schotland J C 2008 Convergence and stability of the inverse scattering series for diffuse waves *Inverse Problems* **24** 065005
[20] Moskow S and Schotland J C 2009 Numerical studies of the inverse Born series for diffuse waves *Inverse Problems* **25** 095007
[21] Romanov V G 1997 Stability estimates in the three-dimensional inverse problem for the transport equation *J. Inverse Ill-Posed Probl.* **5** 46375
[22] Romanov V G 1998 A stability theorem in the problem of the joint determination of the attenuation coefficient and the scattering indicatrix for the stationary transport equation *Mat. Tr.* **1** 78
[23] Schuster T, Kaltenbacher B, Hofmann B and Kazimierski K 2012 *Regularization Methods in Banach Spaces* (Berlin: De Gruyter)
[24] Bonesky T, Kazimierski K S, Maa P, Schöpfer F and Schuster T 2008 Minimization of Tikhonov functionals in Banach spaces *Abstr. Appl. Anal.* **2008** 1–20
[25] Stefanov P 2003 Inverse problems in transport theory *Inside Out: Inverse Problems and Applications* (*MSRI Publications* ed G Uhlmann 47)(Cambridge: Cambridge University Press)
[26] Stefanov P and Tamasan A 2009 Uniqueness and non-uniqueness in inverse radiative transfer *Proc. Am. Math. Soc.* **137** 2335
[27] Stefanov P and Uhlmann G 2003 Optical tomography in two dimensions *Methods Appl. Anal.* **10** 1
[28] Tamasan A 2002 An inverse boundary value problem in two-dimensional transport *Inverse Problems* **18** 209

[29] McDowall S, Stefanov P and Tamasan A 2010 Stability of the gauge equivalent classes in inverse stationary transport *Inverse Problems* **26** 025006

[30] Wang J-N 1999 Stability estimates of an inverse problem for the stationary transport equation *Ann. Inst. Henri Poincaré* A **70** 473