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Survival probability and saturation energy in periodically driven quantum chaotic systems

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ABSTRACT

We study characteristics of the steady state of a random-matrix model with periodical pumping, where the energy increase saturates by quantum localization. We study the dynamics by making use of the survival probability. We found that Floquet eigenstates are separated into the localized and extended states, and the former governs the dynamics.

1. Introduction

Energy dissipation in quantum systems has been studied for a long time. A typical example in one-body systems is the kicked rotator model [1]. In particular, quantum localization is interesting in such dynamics; unlike classical diffusion, the average energy starting initially from the ground state saturates at a finite value after the transient diffusive time-evolution because of quantum interference [2]. That is, the system cannot absorb energy thereafter. The kicked rotator model has been experimentally realized using hydrogen and sodium atoms, and this saturation is observed [3–7]. A spin model [8] was also used to study dynamics of many-body quantum systems. In addition, the energy diffusion of time-dependent random-matrix Hamiltonians has been studied. Wilkinson clarified the dependence of the diffusion constant on symmetries of random matrices [9,10]. Bulgac et al. derived an analytical expression of the diffusion coefficient [11]. Cohen and Kottos studied the diffusion coefficient under an oscillating perturbation [12].

In this Letter, we study the distribution of energy of a random-matrix Hamiltonian with an oscillating external field in the steady state after quantum localization takes place, while in the previous works the initial dissipation has been focused. We introduce the survival probability as the long-time overlap between the time-evolved state and the initial state. We study the saturation energy in terms of the survival probability. We point out that the survival probability is separated in two parts: the localized part to which the localized Floquet eigenstates contribute and the extended part to which the extended Floquet eigenstates make a contribution.

This Letter is organized as follows. In Section 2, we study the survival probability. In Section 3, we study the saturation energy. Finally in Section 4, we give conclusions.

2. Survival probability

2.1. Definition

Let us introduce an $N \times N$ time-dependent Hamiltonian $\mathcal{H}(t)$ with an oscillating external field of frequency ω. This Hamiltonian is periodic with period $2\pi/\omega$.

$$\mathcal{H}\left(t = \frac{2\pi}{\omega}\right) = \mathcal{H}(t = 0). \tag{1}$$

We write the k-th eigenvalue and eigenstate of the initial Hamiltonian $\mathcal{H}(t=0)$ $(\equiv \mathcal{H}_0)$ as E_k and $|k\rangle$ $(k = 0, 1, \ldots, N-1)$. We take the ground state $|0\rangle$ as the initial state. With the help of the Floquet operator $\mathcal{F}$ (the time-evolution operator for a period) [13], the state $|\psi_n\rangle$ after the n-th period $(t = 2\pi n/\omega)$ is written as

$$|\psi_n\rangle = \mathcal{F}^n|0\rangle. \tag{2}$$

The ν-th eigenvalue and eigenstate of $\mathcal{F}$ are written as $e^{i\phi_\nu}$ and $|\phi_\nu\rangle$. We arrange $\{|\phi_\nu\rangle\}$ such that they satisfy $|\langle\phi_{\nu+1}|0\rangle|^2 \leqslant$

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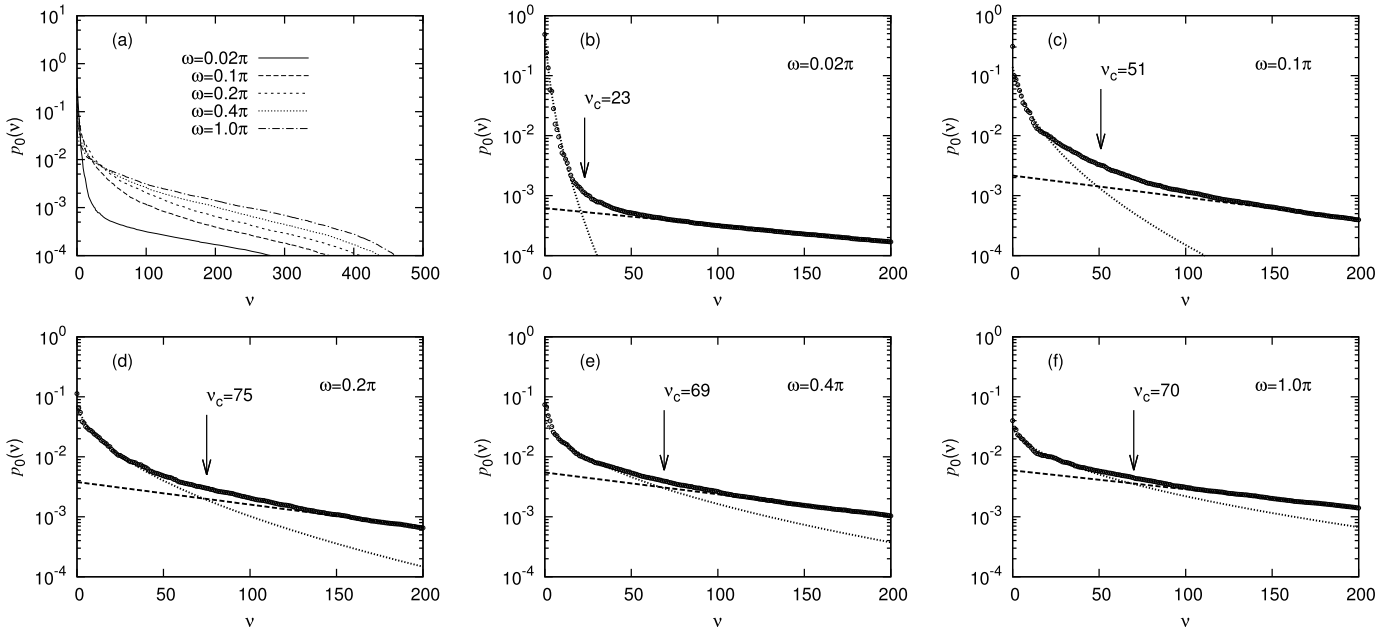

Fig. 1. (a) The five lines show, from the bottom, $p_0(\nu)$ for $\omega = 0.02\pi$, 0.1π, 0.2π, 0.4π, and 1.0π, respectively. The data are obtained by averaging five independent samples. Fittings are shown for $\omega =$ (b) 0.02π, (c) 0.1π, (d) 0.2π, (e) 0.4π, and (f) 1.0π, respectively.

$|\langle\phi_\nu|0\rangle|^2$. We define $p_k(\nu)$ as the overlap between the energy eigenstate $|k\rangle$ and Floquet eigenstate $|\phi_\nu\rangle$:

$$p_k(\nu) \equiv \left|\langle\phi_\nu|k\rangle\right|^2. \tag{3}$$

For a sequence X_n, we define $\overline{X_n}$ as the long-time average: $\overline{X_n} \equiv \lim_{m\to\infty}\sum_{n=0}^{m-1} X_n/m$. We define the survival probability $P(\omega)$ as follows:

$$P(\omega) \equiv \lim_{m\to\infty}\frac{1}{m}\sum_{n=0}^{m-1}\left|\langle 0|\psi_n\rangle\right|^2 = \sum_{\nu,\nu'}\overline{e^{i\alpha(\phi_\nu-\phi_{\nu'})}}\,p_0(\nu)p_0(\nu'). \tag{4}$$

The survival probability has been considered, for example, from the viewpoint of quantum irreversibility [14,15]. In this Letter, the initial state is always taken in the ground state $|0\rangle$. Correspondingly, the survival probability (4) is defined using $|0\rangle$. Therefore, the survival probability reduces only when the transition to upper levels occurs. Moreover, $P(\omega)$ in Eq. (4) is defined as the probability in the long-time limit.

By assuming phases are random, we obtain

$$\overline{e^{i\alpha(\phi_\nu-\phi_{\nu'})}} = \delta_{\nu,\nu'}, \tag{5}$$

and then

$$P(\omega) \simeq \sum_\nu p_0^2(\nu). \tag{6}$$

We count how many Floquet eigenstates are involved in the ground state as a function of ω. For a given value r (<1), we define $f(\omega;r)$ as the minimum number of the Floquet states which satisfy $\sum_{\nu=0}^{N-1} p_0(\nu) > r$ [16,17];

$$f(\omega;r) = \frac{1}{N}\min\left\{N' : \sum_{\nu=0}^{N'-1} p_0(\nu) > r\right\}. \tag{7}$$

2.2. Localized and extended states

To study the numerical behavior of $P(\omega)$, we use the following random-matrix Hamiltonian:

$$\mathcal{H}\big(\lambda(t)\big) = \mathcal{H}_0 + \lambda(t)\mathcal{V}, \quad \lambda(t) = \lambda_0\sin(\omega t), \tag{8}$$

where $\mathcal{H}_0$ and $\mathcal{V}$ are random matrices with $N = 500$ drawn from the Gaussian orthogonal ensemble. We put $\lambda_0 = 0.5$. The variance of the off-diagonal elements of the random matrices is unity. We scale the eigenvalues of $\mathcal{H}_0$ so that they distribute from -1 to 1, i.e., $E_0 = -1$ and $E_{N-1} = 1$.

For quantum chaotic systems, whose level statistics is well described by random matrices, the whole shape of the spectrum is different from the semi-circle of random matrices. Near the ground state $|0\rangle$ of the Hamiltonian (8), the mean level spacing Δ quickly changes. Although the whole shape of the spectrum is not universal, such a quick change is usually seen in quantum chaotic systems (e.g., the two-dimensional well model in [15]). In this Letter, we are interested in the dynamics when Δ is not necessarily constant.

In Fig. 1(a), we show $p_0(\nu)$ as a function of ν for different values of ω. We find that $p_0(\nu)$ is fitted by two functions;

$$p_0(\nu) \simeq \begin{cases} a_{\mathrm{L}}e^{-b_{\mathrm{L}}\sqrt{\nu}} & \text{for small } \nu, \\ a_{\mathrm{E}}e^{-b_{\mathrm{E}}\nu} & \text{for large } \nu, \end{cases} \tag{9}$$

where a_{L}, b_{L}, a_{E}, and b_{E} are positive constants. Figs. 1(b)–(f) show fitting for $\omega = 0.02\pi$, 0.1π, 0.2π, 0.4π, and 1.0π. For small ν, $0 \leqslant \nu \leqslant 50$ are used (first a few singular values are dropped if any). For large ν, $150 \leqslant \nu \leqslant 350$ are used. We define ν_{c} as the closest integer to the real number which satisfies

$$a_{\mathrm{L}}e^{-b_{\mathrm{L}}\sqrt{\nu}} = a_{\mathrm{E}}e^{-b_{\mathrm{E}}\nu}. \tag{10}$$

The values of ν_{c} are shown in Figs. 1(b)–(f).

By using ν_{c}, we can divide $\{|\phi_\nu\rangle\}$ into two groups. We call the states with ν smaller than ν_{c} the *localized* states and the states with ν greater than or equal to ν_{c} the *extended* states:

$$\begin{aligned} &\text{localized states} = \left\{|\phi_\nu\rangle\colon\ \nu < \nu_{\mathrm{c}}\right\}, \\ &\text{extended states} = \left\{|\phi_\nu\rangle\colon\ \nu \geqslant \nu_{\mathrm{c}}\right\}. \end{aligned} \tag{11}$$

Note that $p_0(\nu) > p_0(\nu_{\mathrm{c}})$ for the localized states and $p_0(\nu) \leqslant p_0(\nu_{\mathrm{c}})$ for the extended states. For every ω, $p_0(\nu)$ first drops sharply (localized states) and then has a tail (extended states).

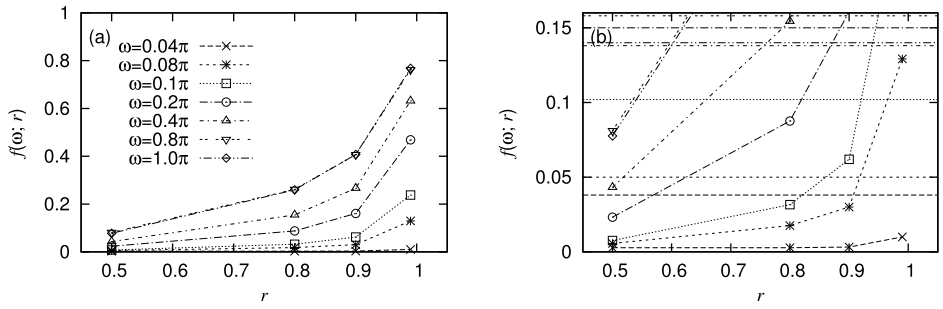

Fig. 3. (a) $f(\omega; r)$ as a function of r for $\omega = 0.04\pi$, 0.08π, 0.1π, 0.2π, 0.4π, 0.8π, and 1.0π. (b) Same as (a) but small values of $f(\omega; r)$ are focused together with horizontal lines which show the values of ν_c/N.

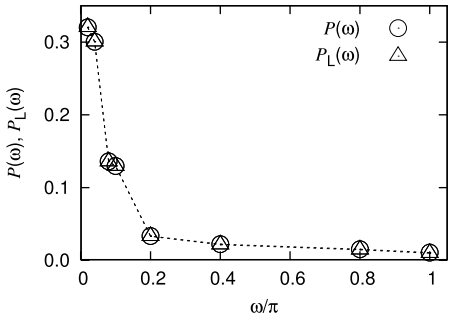

Fig. 2. $P_L(\omega)$ is compared to $P(\omega)$.

Let us similarly separate $P(\omega)$ into two terms according to the localized and extended states:

$$P(\omega) \simeq \sum_{\nu} p_0^2(\nu) = P_L(\omega) + P_E(\omega), \tag{12}$$

where we defined the localized survival probability $P_L(\omega)$ and the extended survival probability $P_E(\omega)$ as

$$P_L(\omega) = \sum_{\nu<\nu_c} p_0^2(\nu), \qquad P_E(\omega) = \sum_{\nu\geqslant\nu_c} p_0^2(\nu). \tag{13}$$

Fig. 2 shows $P_L(\omega)$ as a function of ω together with $P(\omega)$. We see that the dominant behavior of $P(\omega)$ is governed by $P_L(\omega)$ even though only a small fraction of the Floquet states contribute to $P_L(\omega)$.

Fig. 3(a) shows the number of relevant Floquet eigenstates $f(\omega; r)$ as a function of r for $\omega = 0.04\pi$ ($\times$), 0.08π ($*$), 0.1π ($\square$), 0.2π ($\bigcirc$), 0.4π ($\triangle$), 0.8π ($\triangledown$), and 1.0π ($\diamond$). In the figure, each mark is obtained as an average over five samples. It is found that, for large ω, more than half of the Floquet eigenstates are needed to cover the initial state $|0\rangle$ when r is larger than 0.9. This means that the contribution of the extended states $|\phi_\nu\rangle$ ($\nu \geqslant \nu_c$) becomes more important in this regime. The behavior of $f(\omega; r)$ for smaller r, however, implies that the localized states $|\phi_\nu\rangle$ ($\nu < \nu_c$) govern $P(\omega)$ even for large ω. To investigate how many extended states contribute, small values of $f(\omega; r)$ are focused in Fig. 3(b). In the figure, fractions of the localized states (ν_c/N) are also plotted. We have $f(\omega; 0.5) < \frac{\nu_c}{N}$ for every ω. We have $f(\omega; 0.8) < \frac{\nu_c}{N}$ for $\omega \leqslant 0.2\pi$, $f(\omega; 0.9) < \frac{\nu_c}{N}$ for $\omega \leqslant 0.1\pi$, and $f(\omega; 0.99) < \frac{\nu_c}{N}$ for $\omega \leqslant 0.04\pi$.

3. Saturation energy

The energy expectation value $E(n, \omega)$ at the n-th period is calculated as

$$\begin{aligned} E(n,\omega) &= \langle\psi_n|\mathcal{H}\left(t = \frac{2\pi n}{\omega}\right)|\psi_n\rangle = \langle\psi_n|\mathcal{H}_0|\psi_n\rangle \\ &= \langle 0|\left(\mathcal{F}^n\right)^\dagger \mathcal{H}_0 \mathcal{F}^n|0\rangle \\ &= \sum_{k=0}^{N-1} E_k \sum_{\nu,\nu'} e^{in(\phi_\nu-\phi_{\nu'})}\langle 0|\phi_{\nu'}\rangle\langle\phi_{\nu'}|k\rangle\langle k|\phi_\nu\rangle\langle\phi_\nu|0\rangle. \end{aligned} \tag{14}$$

Let us consider the saturation energy $E_{sat}(\omega)$, which is the long-time average of $E(n, \omega)$ [18].

$$\begin{aligned} E_{sat}(\omega) &\equiv \overline{E(n,\omega)} \\ &\equiv \lim_{m\to\infty}\frac{1}{m}\sum_{n=0}^{m-1} E(n,\omega) \\ &= \sum_{k=0}^{N-1} E_k \sum_{\nu,\nu'} \overline{e^{in(\phi_\nu-\phi_{\nu'})}}\langle 0|\phi_{\nu'}\rangle\langle\phi_{\nu'}|k\rangle\langle k|\phi_\nu\rangle\langle\phi_\nu|0\rangle \\ &= \sum_{k=0}^{N-1} E_k \sum_{\nu=0}^{N-1} p_0(\nu)p_k(\nu). \end{aligned} \tag{15}$$

The distributions of $p_2(\nu)$ and $p_9(\nu)$ are shown in Fig. 4 for $\omega = 0.1\pi$ ($k = 2, 9$ are chosen as typical examples). To compare, $p_0(\nu)$ is also shown in the figure. We see that $p_k(\nu)$ is quickly randomized as a function of k, and $p_k(\nu)$ ($k \neq 0$) spreads randomly. When $k \geqslant k_{rand}$ with an integer k_{rand} determined by the system (1) and the initial condition, we can approximate $p_k(\nu)$ as

$$p_k(\nu) \simeq \frac{1}{N} \quad \text{for } k \geqslant k_{rand}. \tag{16}$$

The number k_{rand} is small compared with N but depends on N and also weakly depends on ω.

Let us consider the upper bound and lower bound of E_{sat}:

$$E_{min} \leqslant E_{sat} \leqslant E_{max}. \tag{17}$$

Let us estimate E_{max}. Using Eq. (15), we have

$$E_{sat}(\omega) \leqslant \sum_{k=k_{rand}}^{N-1} E_k \sum_{\nu=0}^{N-1} p_0(\nu)p_k(\nu) \simeq \frac{1}{N}\sum_{k=k_{rand}}^{N-1} E_k. \tag{18}$$

The term $\sum_k E_k/N$ vanishes in the limit $N \to \infty$ because the distribution of $\{E_k\}$ follows the semi-circle law. Therefore, we estimate

$$E_{max} \simeq 0. \tag{19}$$

Next we estimate E_{min}. Using Eq. (15), we have

$$E_{sat}(\omega) \geqslant \sum_{k=0}^{k_{rand}-1} E_k \sum_{\nu=0}^{N-1} p_0^2(\nu) + \sum_{k=k_{rand}}^{N-1} E_k \sum_{\nu=0}^{N-1} p_0(\nu)p_k(\nu). \tag{20}$$

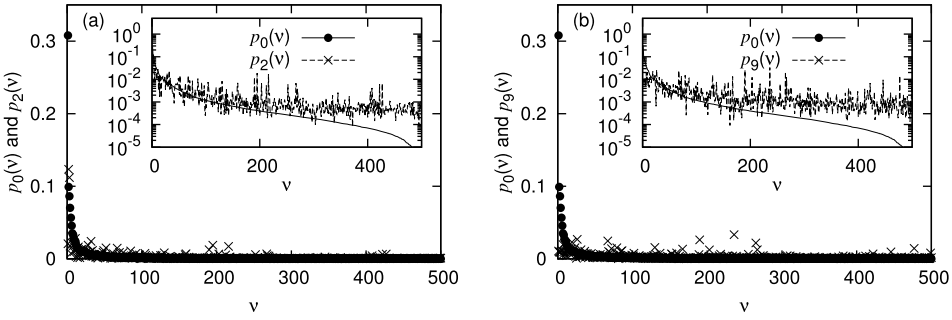

Fig. 4. The distributions of (a) $p_2(\nu)$ and (b) $p_9(\nu)$ are shown for $\omega = 0.1\pi$. The insets show semi-log plots. The data are averaged with five independent samples. For comparison, $p_0(\nu)$ is plotted together.

Thus, we obtain for large N

$$E_{\min} \simeq \sum_{k=0}^{k_{\mathrm{rand}}-1} E_k P(\omega) + \frac{1}{N} \sum_{k=k_{\mathrm{rand}}}^{N-1} E_k \simeq \left(\sum_{k=0}^{k_{\mathrm{rand}}-1} E_k \right) P(\omega). \quad (21)$$

The factor $\sum_k E_k$ is not universal and depends on the system even if the level statistics of the system is described by random matrices.

Finally we consider the variance of the energy $E(n,\omega)$. To estimate the variance, we use the nonresonance assumption [18,19] for the Floquet eigenvalues, which assumes that all the differences of the eigenvalues are different. That is, if $\phi_{\nu_1} - \phi_{\nu_2} = \phi_{\nu_4} - \phi_{\nu_3} \neq 0$ such that $|\langle 0|\phi_{\nu_i}\rangle|^2 \neq 0$ ($i = 1, 2, 3,$ and 4), then $\nu_1 = \nu_4$ and $\nu_2 = \nu_3$. With the help of the nonresonance assumption in addition to relations (5) and (16), we have

$$\begin{aligned} \mathrm{var}\big(E(n,\omega)\big) &\equiv \overline{E^2(n,\omega)} - \big(\overline{E(n,\omega)}\big)^2 \\ &= \sum_{\nu,\nu'} p_0(\nu) p_0\big(\nu'\big) \\ &\quad \times \sum_{k,k'} E_k \langle \phi_{\nu'}|k\rangle \langle k|\phi_\nu\rangle E_{k'} \big\langle \phi_\nu \big| k' \big\rangle \big\langle k' \big| \phi_{\nu'} \big\rangle. \end{aligned} \quad (22)$$

Since $p_0(\nu') \leqslant 1$, we have

$$\begin{aligned} \mathrm{var}\big(E(n,\omega)\big) &\leqslant \sum_k E_k^2 \sum_\nu p_0(\nu) p_k(\nu) \\ &\leqslant \sum_{k=0}^{k_{\mathrm{rand}}-1} E_k^2 \sum_\nu p_0^2(\nu) + \sum_{k=k_{\mathrm{rand}}}^{N-1} E_k^2 \sum_\nu p_0(\nu) p_k(\nu) \\ &\simeq \left(\sum_{k=0}^{k_{\mathrm{rand}}-1} E_k^2 \right) P(\omega) + \frac{1}{N} \sum_{k=k_{\mathrm{rand}}}^{N-1} E_k^2. \end{aligned} \quad (23)$$

Thus, for large N, the variance is estimated as

$$\mathrm{var}\big(E(n,\omega)\big) \leqslant \left(\sum_{k=0}^{k_{\mathrm{rand}}-1} E_k^2 \right) P(\omega). \quad (24)$$

4. Conclusions

We pointed out that the survival probability $P(\omega)$ can be separated into the localized part $P_{\mathrm{L}}(\omega)$ and the extended part $P_{\mathrm{E}}(\omega)$. We further showed that $P_{\mathrm{L}}(\omega)$ is dominant even though relatively small number of states $|\phi_\nu\rangle$ ($\nu < \nu_{\mathrm{c}}$) are involved. The saturation energy $E_{\mathrm{sat}}(\omega)$ and its variance $\mathrm{var}(E(n,\omega))$ are given in terms of the survival probability $P(\omega)$.

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