Survival probability and saturation energy in periodically driven quantum chaotic systems

Manabu Machida\textsuperscript{a,}\textsuperscript{*}, Seiji Miyashita\textsuperscript{b}

\textsuperscript{a} Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA
\textsuperscript{b} Department of Physics, The University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan

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\textbf{A B S T R A C T}

We study characteristics of the steady state of a random-matrix model with periodical pumping, where the energy increase saturates by quantum localization. We study the dynamics by making use of the survival probability. We found that Floquet eigenstates are separated into the localized and extended states, and the former governs the dynamics.

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\section{1. Introduction}

Energy dissipation in quantum systems has been studied for a long time. A typical example in one-body systems is the kicked rotator model \cite{1}. In particular, quantum localization is interesting in such dynamics; unlike classical diffusion, the average energy starting initially from the ground state saturates at a finite value after the transient diffusive time-evolution because of quantum interference \cite{2}. That is, the system cannot absorb energy thereafter. The kicked rotator model has been experimentally realized using hydrogen and sodium atoms, and this saturation is observed \cite{3–7}. A spin model \cite{8} was also used to study dynamics of many-body quantum systems. In addition, the energy diffusion of time-dependent random-matrix Hamiltonians has been studied. Wilkinson clarified the dependence of the diffusion constant on symmetries of random matrices \cite{9,10}. Bulgac et al. derived an analytical expression of the diffusion coefficient \cite{11}. Cohen and Kottos studied the diffusion coefficient under an oscillating perturbation \cite{12}.

In this Letter, we study the distribution of energy of a random-matrix Hamiltonian with an oscillating external field in the steady state after quantum localization takes place, while in the previous works the initial dissipation has been focused. We introduce the survival probability as the long-time overlap between the time-evolved state and the initial state. We study the saturation energy in terms of the survival probability. We point out that the survival probability is separated in two parts: the localized part to which the localized Floquet eigenstates contribute and the extended part to which the extended Floquet eigenstates make a contribution.

This Letter is organized as follows. In Section 2, we study the survival probability. In Section 3, we study the saturation energy. Finally in Section 4, we give conclusions.

\section{2. Survival probability}

\subsection{2.1. Definition}

Let us introduce an $N \times N$ time-dependent Hamiltonian $\mathcal{H}(t)$ with an oscillating external field of frequency $\omega$. This Hamiltonian is periodic with period $2\pi/\omega$.

$$\mathcal{H}(t = \frac{2\pi}{\omega}) = \mathcal{H}(t = 0).$$

(1)

We write the $k$-th eigenvalue and eigenstate of the initial Hamiltonian $\mathcal{H}(t = 0)$ as $E_k$ and $|k\rangle$ ($k = 0, 1, \ldots, N - 1$). We take the ground state $|0\rangle$ as the initial state. With the help of the Floquet operator $\mathcal{F}$ (the time-evolution operator for a period) \cite{13}, the state $|\psi_n\rangle$ after the $n$-th period ($t = 2\pi n/\omega$) is written as

$$|\psi_n\rangle = \mathcal{F}^n|0\rangle.$$ 

(2)

The $\nu$-th eigenvalue and eigenstate of $\mathcal{F}$ are written as $e^{i\phi_\nu}$ and $|\phi_\nu\rangle$. We arrange $|\langle \phi_{\nu+1} |0\rangle|^2 \leq$$
2.2. Localized and extended states

We define $p_k(v)$ as the overlap between the energy eigenstate $|k\rangle$ and Floquet eigenstate $|\phi_v\rangle$:

$$p_k(v) \equiv |\langle \phi_v | k \rangle|^2.$$  \hfill (3)

For a sequence $X_n$, we define $\overline{X_n}$ as the long-time average:

$$\overline{X_n} = \lim_{m \to \infty} \frac{1}{m} \sum_{n=0}^{m-1} X_n.$$  \hfill (4)

The survival probability has been considered, for example, from the viewpoint of quantum irreversibility [14,15]. In this Letter, the initial state is always taken in the ground state $|0\rangle$. Correspondingly, the survival probability (4) is defined using $|0\rangle$. Therefore, the survival probability reduces only when the transition to upper levels occurs. Moreover, $P(\omega)$ in Eq. (4) is defined as the probability in the long-time limit.

By assuming phases are random, we obtain

$$e^{i\langle \phi_v | \phi_{v'} \rangle} = \delta_{v,v'},$$  \hfill (5)

and then

$$P(\omega) \geq \sum_v p^2_{0}(v).$$  \hfill (6)

We count how many Floquet eigenstates are involved in the ground state as a function of $\omega$. For a given value $r (<1)$, we define $f(\omega; r)$ as the minimum number of the Floquet states which satisfy $\sum_{v=0}^{r-1} p_0(v) > r$ [16,17]:

$$f(\omega; r) = \frac{1}{N} \min \left\{ N' : \sum_{v=0}^{N'} p_0(v) > r \right\}. $$  \hfill (7)

2.2. Localized and extended states

To study the numerical behavior of $P(\omega)$, we use the following random-matrix Hamiltonian:

$$\mathcal{H}(\lambda(t)) = \mathcal{H}_0 + \lambda(t) V, \quad \lambda(t) = \lambda_0 \sin(\omega t),$$  \hfill (8)

where $\mathcal{H}_0$ and $V$ are random matrices with $N = 500$ drawn from the Gaussian orthogonal ensemble. We put $\lambda_0 = 0.5$. The variance of the off-diagonal elements of the random matrices is unity. We scale the eigenvalues of $\mathcal{H}_0$ so that they distribute from $-1$ to $1$, i.e., $E_0 = -1$ and $E_{N-1} = 1$.

For quantum chaotic systems, whose level statistics is well described by random matrices, the whole shape of the spectrum is different from the semi-circle of random matrices. Near the ground state $|0\rangle$ of the Hamiltonian (8), the mean level spacing $\Delta$ quickly changes. Although the whole shape of the spectrum is not universal, such a quick change is usually seen in quantum chaotic systems (e.g., the two-dimensional well model in [15]). In this Letter, we are interested in the dynamics when $\Delta$ is not necessarily constant.

In Fig. 1(a), we show $p_0(v)$ as a function of $v$ for different values of $\omega$. We find that $p_0(v)$ is fitted by two functions:

$$p_0(v) \simeq \begin{cases} a_1 e^{-b_1 \sqrt{v}} & \text{for small } v, \\ a_2 e^{-b_2 v} & \text{for large } v, \end{cases}$$  \hfill (9)

where $a_1$, $b_1$, $a_2$, and $b_2$ are positive constants. Figs. 1(b)–(f) show fitting for $\omega = 0.02\pi$, $0.1\pi$, $0.2\pi$, $0.4\pi$, and $1.0\pi$. For small $v$, $0 \leq v \leq 50$ are used (first a few singular values are dropped if any). For large $v$, $150 \leq v \leq 350$ are used. We define $v_c$ as the closest integer to the real number which satisfies

$$a_1 e^{-b_1 \sqrt{v}} = a_2 e^{-b_2 v}.$$  \hfill (10)

The values of $v_c$ are shown in Figs. 1(b)–(f).

By using $v_c$, we can divide $|\phi_{v_0}\rangle$ into two groups. We call the states with $v$ smaller than $v_c$ the localized states and the states with $v$ greater than or equal to $v_c$ the extended states:

localized states $= \left\{ |\phi_{v}\rangle : v < v_c \right\}$,  

extended states $= \left\{ |\phi_{v}\rangle : v \geq v_c \right\}$.  \hfill (11)

Note that $p_0(v) > p_0(v_c)$ for the localized states and $p_0(v) \leq p_0(v_c)$ for the extended states. For every $\omega$, $p_0(v)$ first drops sharply (localized states) and then has a tail (extended states).
In the figure, fractions of the localized states (f states contribute, small values of r mark is obtained as an average over five samples. It is found that, for r ⩽ 0.1π, the dominant behavior of P r is governed by the extended states |ψ ν ⟩ |ψ ν ⟩, though only a small fraction of the Floquet states contribute to P r. We have calculated as

\[ P(ω) = \sum_v p^2_0(v) = P_L(ω) + P_E(ω), \tag{12} \]

where we defined the localized survival probability P_L(ω) and the extended survival probability P_E(ω) as

\[ P_L(ω) = \sum_{v < ν_c} p^2_0(v), \quad P_E(ω) = \sum_{v ≥ ν_c} p^2_0(v). \tag{13} \]

Fig. 2 shows P_L(ω) as a function of ω together with P(ω). We see that the dominant behavior of P(ω) is governed by P_L(ω) even though only a small fraction of the Floquet states contribute to P_L(ω).

Fig. 3(a) shows the number of relevant Floquet eigenstates f(ω; r) as a function of r for ω = 0.04π (△), 0.08π ( ), 0.1π (□), 0.2π (○), 0.4π (△), 0.8π (□), 1.0π (○), in the figure, each mark is obtained as an average over five samples. It is found that, for large ω, more than half of the Floquet eigenstates are needed to cover the initial state |0⟩ when r is larger than 0.9. This means that the contribution of the extended states |ψ ν ⟩ (v < ν_c) becomes more important in this regime. The behavior of f(ω; r) for smaller r, however, implies that the localized states |ψ ν ⟩ (v ≥ ν_c) govern P(ω) even for large ω. To investigate how many extended states contribute, small values of f(ω; r) are focused in Fig. 3(b).

The distributions of p_2(v) and p_9(v) are shown in Fig. 4 for ω = 0.1π (k = 2.9 are chosen as typical examples). To compare, p_9(v) is also shown in the figure. We see that p_9(v) is quickly randomized as a function of k, and p_2(v) (k ≠ 0) spreads randomly. When k ⩾ k_{rand} with an integer k_{rand} determined by the system (1) and the initial condition, we can approximate p_k(v) as

\[ p_k(v) ≃ \frac{1}{N} \text{ for } k ⩾ k_{rand}. \tag{16} \]

The number k_{rand} is small compared with N but depends on N and also weakly depends on ω.

Let us consider the upper bound and lower bound of E_{sat}:

\[ E_{min} ≤ E_{sat} ≤ E_{max}. \tag{17} \]

Let us estimate E_{max}. Using Eq. (15), we have

\[ E_{sat}(ω) ≤ \sum_{k = k_{rand}}^{N-1} \frac{N-1}{N} E_k(p_0(v) p_k(v) ≃ \frac{1}{N} \sum_{k = k_{rand}}^{N-1} E_k. \tag{18} \]

The term \[ \sum_k E_k/N \] vanishes in the limit N → ∞ because the distribution of \[ E_k \] follows the semi-circle law. Therefore, we estimate

\[ E_{max} ≃ 0. \tag{19} \]

Next we estimate E_{min}. Using Eq. (15), we have

\[ E_{sat}(ω) ≥ \sum_{k = 0}^{k_{rand}-1} \frac{N-1}{N} p_0(v) p_k(v) + \sum_{k = k_{rand}}^{N-1} E_k. \tag{20} \]

3. Saturation energy

The energy expectation value E(n, ω) at the n-th period is calculated as

\[ E(n, ω) = ⟨ψ_n|H(t = \frac{2πn}{ω})|ψ_n⟩ = ⟨ψ_n|H_0|ψ_n⟩ = 0 \text{ for } 0 < 0.08π \text{ for } ω < 0.1π, \text{ and } f(ω; 0.99) < \frac{1}{N} \text{ for } ω ≤ 0.04π. \]

\[ E(n, ω) = \frac{1}{N} \sum_{k = 0}^{N-1} E_k. \tag{14} \]

Let us consider the saturation energy E_{sat}(ω), which is the long-time average of E(n, ω) [18].

\[ E_{sat}(ω) = \lim_{m → ∞} \frac{1}{m} \sum_{n = 0}^{m-1} E(n, ω) \]

\[ E_{sat}(ω) = \sum_{k = 0}^{N-1} \frac{N-1}{N} E_k(p_0(v) p_k(v)). \tag{15} \]
Thus, we obtain for large $N$
\[
E_{\text{min}} \simeq \sum_{k=0}^{k_{\text{rand}}-1} E_k P(\omega) + \frac{1}{N} \sum_{k=k_{\text{rand}}}^{N-1} E_k \simeq \left( \sum_{k=0}^{k_{\text{rand}}-1} E_k \right) P(\omega). \tag{21}
\]

The factor $\sum_k E_k$ is not universal and depends on the system even if the level statistics of the system is described by random matrices.

Finally we consider the variance of the energy $E(n, \omega)$. To estimate the variance, we use the nonresonance assumption in addition to relations (5) and (16), we have
\[
\text{var}(E(n, \omega)) = \overline{E^2(n, \omega)} - (\overline{E(n, \omega)})^2
\]
\[
= \sum_{v, v'} p_0(v)p_0(v')
\]
\[
\times \sum_{k,k'} E_k \langle \phi_{0,v'}|k\rangle\langle k|\phi_{0,v}\rangle E_{k'} \langle \phi_{0,v'}|k'\rangle\langle k'|\phi_{0,v'}\rangle.	ag{22}
\]

Since $p_0(v') \leq 1$, we have
\[
\text{var}(E(n, \omega)) \leq \sum_k E_k^2 \sum_v p_0(v)p_k(v)
\]
\[
\leq \sum_{k=0}^{k_{\text{rand}}-1} E_k^2 \sum_v p_0(v) + \sum_{k=k_{\text{rand}}}^{N-1} E_k^2 \sum_v p_0(v)p_k(v)
\]
\[
\simeq \left( \sum_{k=0}^{k_{\text{rand}}-1} E_k^2 \right) P(\omega) + \frac{1}{N} \sum_{k=k_{\text{rand}}}^{N-1} E_k^2. \tag{23}
\]

Thus, for large $N$, the variance is estimated as
\[
\text{var}(E(n, \omega)) \leq \left( \sum_{k=0}^{k_{\text{rand}}-1} E_k^2 \right) P(\omega). \tag{24}
\]

4. Conclusions

We pointed out that the survival probability $P(\omega)$ can be separated into the localized part $P_1(\omega)$ and the extended part $P_2(\omega)$. We further showed that $P_1(\omega)$ is dominant even though relatively small number of states $|\phi_{0,v}\rangle$ ($v < v_c$) are involved. The saturation energy $E_{\text{sat}}(\omega)$ and its variance $\text{var}(E(n, \omega))$ are given in terms of the survival probability $P(\omega)$.

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