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Frequency Dependence of Quantum Localization in a Periodically Driven System

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Under a periodically driven external field or a temporally oscillating boundary, quantum few-body systems whose corresponding classical systems show chaos are modeled by the random matrices with an oscillating parameter. In such systems, time evolution of the system from the ground state shows the quantum diffusion. However, due to periodic oscillation of the parameter, quantum interference effect prevents the energy of the system from diverging and the energy saturates after some transient time. We show an example where the energy eventually fluctuates around a constant saturated energy. Since the value of this saturated energy depends on the frequency of the driving, we also investigate the dependence of the saturated energy on the frequency in the small frequency region.

KEYWORDS: quantum dynamics, quantum localization, random matrix

Under periodic driving force, the initial state which is the ground state eventually settles down in a state corresponding to a finite temperature. This saturation of the energy is a well known phenomenon, but its dependence on the frequency of the periodic force is still an open problem. We study such evolution in a quantum few-body system, and uncover the dependence of the saturation energy on the frequency. This driving could be an oscillating external magnetic field acting on a quantum spin system, an oscillating boundary of quantum billiards, or so. Under such driving forces, saturation of the energy occurs after some transient time. If interactions of the system are strong or the boundary of a billiard has less symmetry, dynamics of the corresponding classical systems becomes chaos, and Hamiltonians of the systems can be represented by random matrices.[1] Hence, the random matrix theory gives a powerful tool to study quantum systems which have chaotic nature not only for their static characters, such as the spectral statistics, but also for their dynamics,[2–4] although the correspondence between random matrices and Hamiltonians of realistic systems is not trivial.[5] Using a random matrix model with a sinusoidally changing parameter, we present a model where the energy eventually saturates and fluctuates around a constant saturated energy. We investigate how the saturated energy depends on the frequency of the driving field.

The random matrix model we shall study is written as

$$H(t) = H_0 + \lambda(t)V. \tag{1}$$

Here H_0 and V are random matrices of dimension N independently drawn from Gaussian Orthogonal Ensemble (GOE). The elements satisfy $\langle (H_0)_{ij}\rangle = 0$, $\langle (H_0)_{ij}^2\rangle = 1+\delta_{ij}$, $\langle (V)_{ij}\rangle = 0$, $\langle (V)_{ij}^2\rangle = 1+\delta_{ij}$. The parameter $\lambda(t)$ denotes a periodic driving with the frequency ω: $\lambda(t) = A\sin(\omega t)$, $A = 0.5$.

Let the initial state $|\psi_0\rangle$ at $t=0$ be the ground state of H_0. We define the Floquet operator F whose eigenstate is $|\nu\rangle$, with the eigenvalue $\mathrm{e}^{\mathrm{i}\phi_\nu}$;

$$F = T\exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^{2\pi/\omega} H(t)\,\mathrm{d}t\right], \tag{2a}$$

$$F|\nu\rangle = \mathrm{e}^{\mathrm{i}\phi_\nu}|\nu\rangle \quad (\nu = 1,\ldots,N), \tag{2b}$$

where T denotes the time-ordered product and we arrange ν so that $|\langle\nu|\psi_0\rangle|^2 > |\langle\nu'|\psi_0\rangle|^2$ for $\nu > \nu'$. Then, the state $|\psi_n\rangle$ after the n-th period is written as

$$|\psi_n\rangle = F^n|\psi_0\rangle. \tag{3}$$

Using this fact, the expectation value $E(n)$ of energy after the n-th period can be expanded by the Floquet eigenstates;

$$\begin{aligned} E(n) &= \langle\psi_n|H_0|\psi_n\rangle \\ &= \sum_{\nu,\nu'} \mathrm{e}^{\mathrm{i}n(\phi_{\nu'}-\phi_\nu)}\langle\psi_0|\nu\rangle\langle\nu|H_0|\nu'\rangle\langle\nu'|\psi_0\rangle. \end{aligned} \tag{4}$$

Now, we shall show $E(n)$ fluctuates around a constant energy E_{sat}, which depends on the frequency ω, when n is sufficiently large. The time average of $E(n)$ gives the saturated energy E_{sat}.

$$\begin{aligned} \overline{E(n)} &\equiv \lim_{m\to\infty}\frac{1}{m}\sum_{n=0}^{m-1} E(n) \\ &= \sum_\nu |\langle\nu|\psi_0\rangle|^2\langle\nu|H_0|\nu\rangle \\ &\equiv E_{\mathrm{sat}}, \end{aligned} \tag{5}$$

where we use the relation $\overline{\mathrm{e}^{\mathrm{i}n(\phi_{\nu'}-\phi_\nu)}} = \delta_{\nu,\nu'}$. The variance can be estimated as

$$\begin{aligned} \mathrm{var}[E(n)] &= \overline{\left(E(n)-\overline{E(n)}\right)^2} = \overline{E^2(n)} - E_{\mathrm{sat}}^2 \\ &\le ||H_0||^2, \quad ||H_0|| \equiv \max_{j,j'}|(H_0)_{jj'}|, \end{aligned} \tag{6}$$

where we assume the nonresonance condition that all the differences of the eigenvalues are different, that is, if $\phi_{\nu_1} - \phi_{\nu_2} = \phi_{\nu_4} - \phi_{\nu_3} \neq 0$ such that $|\langle\nu_i|\psi_0\rangle|^2 \neq$

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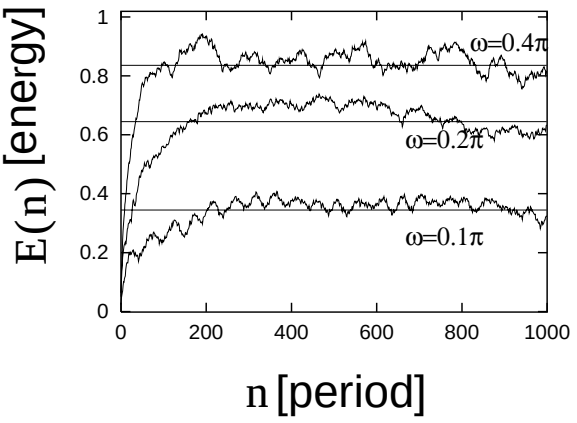

Fig. 1. Time evolution of $E(n)$ is shown with the corresponding $E_{\rm sat}$ (horizontal line). $N = 700$ and $\omega = 0.1\pi$, 0.2π and 0.4π, respectively.

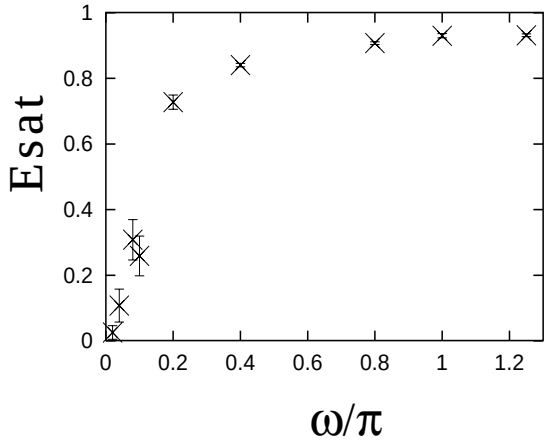

Fig. 2. $E_{\rm sat}$ as a function of ω/π. $N = 700$. Each point is obtained from being averaged over five samples. Error bars show the variance.

0, $i = 1, 2, 3$ and 4, then $\nu_1 = \nu_4$, $\nu_2 = \nu_3$. This condition is plausible because H_0 is a random matrix and $\mathrm{e}^{\mathrm{i}\phi_\nu}$ distributes randomly on a unit circle. Equation (6) implies that $E(n)$ does not deviate far from $E_{\rm sat}$.

Figure 1 shows time evolution of the average energy, $E(n)$, for three values of ω in the case of $N = 700$. Indeed, we see that $E(n)$ fluctuates around $E_{\rm sat}$. The energy level distribution has a symmetric shape about the center of the spectrum. We remind that an energy unit is so determined that the ground state energy is zero and the center of the spectrum is one.

In Fig. 1, $E_{\rm sat}$ clearly depends on the frequency ω. We shall focus on the saturated energy $E_{\rm sat}$. In order to see its dependence on the frequency, we study five samples, and plot the average of $E_{\rm sat}$ as a function of ω in Fig. 2. There, $N = 700$ and error bars show variances of $E_{\rm sat}$.

Here we estimate the effective pumping rate $p(k)$ from the ground state to an excited state $|k\rangle$ [$E_{\rm sat} = \sum_k p(k)E_k$]. Here, E_k is the k-th eigenvalue of H_0 and $|k\rangle$ is the eigenstate of H_0 corresponding to E_k. Generally, this rate is given by[6] $\sum_\nu |\langle\nu|\psi_0\rangle|^2|\langle\nu|k\rangle|^2$. Let ℓ be the localization length in the Floquet space:

$$|\langle\nu|\psi_0\rangle|^2 \propto \mathrm{e}^{-\nu/\ell}. \tag{7}$$

In addition, it has been suggested,[7]

$$|\langle\nu|k\rangle|^2 \sim \mathrm{e}^{-|\nu-k|/\ell}. \tag{8}$$

Thus, we have

$$p(k) = \sum_\nu |\langle\nu|\psi_0\rangle|^2|\langle\nu|k\rangle|^2 \sim \mathrm{e}^{-k/\ell}. \tag{9}$$

We have pointed out that ℓ is proportional to ω[6] in the region where the scattering process consists of successive transitions of the state to adjacent levels at avoided level crossing points (the Landau–Zener region). Thus we conclude that in this region (we numerically checked $\omega \simeq 0.1\pi$ is in the Landau–Zener region) $p(k)$ is expressed as

$$p(k) \sim \mathrm{e}^{-\alpha k/\omega}, \tag{10}$$

where α is a constant.

This conclusion means that, in the quantum localization, the pumping rate in the saturated state depends on ω exponentially, which is not compatible to the picture of independent Landau–Zener transition[8,9] at avoided level crossing points.[6] This picture ignores the quantum phase interference and gives the result $E_{\rm sat} \to 1$ regardless of ω. The ω dependence of $p(k)$ in eq. (10) leads that $E_{\rm sat} \to 0$ at very small ω. In addition, eq. (10) implies that the saturated value $E_{\rm sat}$ is always below 1. This is qualitatively consistent with the ω dependence of $E_{\rm sat}$ in Fig. 2 at small ω. Quantitative comparison between $E_{\rm sat}$ calculated using eq. (10) and the numerical data is a future issue, since it needs more data points in Fig. 2 and for example, in eq. (10), α is an unknown amplitude. In short, to have the form as shown in Fig. 2, the quantum phase interference is essential.

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