A Brief Introduction to Inverse Problems by Carleman Estimates *

Manabu Machida

December 1, 2011

1 Introduction and History

The objective of this lecture is (i) to give a brief introduction to inverse problems by Carleman estimates and (ii) to provide a minimal set of terminology and techniques which are necessary to read research papers in this field. The lecture assumes basic calculus but advanced materials will be explained in class.

A Carleman estimate was first used to prove uniqueness for the Cauchy problem with data on a non-characteristic curve [1, 2]. The technique was then imported to the field of inverse problems as a tool to determine coefficients of a partial differential equation from boundary values [3]. Since then, inverse problems by Carleman estimates have been intensively studied. See, for example, [4] and [5] for Carleman estimates for hyperbolic equations. See a recent review [6] for Carleman estimates for parabolic equations. The textbook by Isakov [7] is also a good place to start.

MM learned Carleman estimates on a graduate course taught by Prof. Masahiro Yamamoto at The University of Tokyo [8]. This lecture note is based on Prof. Yamamoto's lecture. Here, Carleman estimates will be explained by focusing on an inverse problem for a simple hyperbolic equation.

2 X-ray in Biological Tissue

Let us consider the transport of X-rays in biological tissue. Since X-rays penetrate the medium without scattering,[1] it propagates along a line (say, the x-axis). The X-ray enters the medium at $x = 0$, is partially absorbed, and exits at $x = \ell$. This transport phenomenon is modeled by the following transport equation in a one-dimensional random medium with the length ℓ (see [9] for intuitive derivation of the transport equation and see [10] for rigorous derivation from the Maxwell equations). In the equation, $u(x,t) \in \mathbb{R}$ is the intensity of the X-ray at position $x \in (0, \ell)$ and time $t \in (-T, T)$, and $p(x) \in \mathbb{R}$ is the absorption coefficient.

$$\begin{cases} \dfrac{\partial}{\partial t}u(x,t) + \dfrac{\partial}{\partial x}u(x,t) + p(x)u(x,t) = 0, \\ u(x,0) = a(x), \quad 0 < x < \ell, \\ u(0,t) = b(t), \quad -T < t < T. \end{cases} \tag{2.1}$$

*I gave a 2×1.5-hour lecture in "Methods of Applied Mathematics I: Applied Functional Analysis" (Math 556) at University of Michigan.

[1] In reality, X-rays are sometimes scattered, but we can ignore such scatterings to a good approximation.

Here $a(x)$ (>0) and $b(t)$ are the initial and boundary conditions. Let us assume that $p \in L^{\infty}(0,\ell)$ is unknown. We want to determine $p(x)$ by the measured boundary value $u(\ell,t)$. Our goal is to establish a stability estimate such as

$$||p-q||_{L^2(0,\ell)} \leq C||u[p](\ell,t)-u[q](\ell,t)||_* \tag{2.2}$$

with a suitable norm $||\cdot||_*$ for two different absorption coefficients p and q. At the end of the day, we will prove Theorem 5.1. Throughout this note, we let C denote a generic positive constant.

3 Linearization

Let us put

$$\tilde{u}(x,t)=u[q](x,t)-u[p](x,t), \quad f(x)=p(x)-q(x), \quad R(x,t)=u[q](x,t). \tag{3.1}$$

We then have

$$\begin{cases} \dfrac{\partial}{\partial t}\tilde{u}(x,t)+\dfrac{\partial}{\partial x}\tilde{u}(x,t)+p(x)\tilde{u}(x,t)=f(x)R(x,t), \\ \tilde{u}(x,0)=0, \quad 0<x<\ell, \\ \tilde{u}(0,t)=0, \quad -T<t<T. \end{cases} \tag{3.2}$$

We assume that $q(x)$ is known. Our job is to estimate $f(x)$ in the source term. As we will see later, the initial condition must be nonzero. To have a nonzero initial condition, let us differentiate the equation with respect to t. We put

$$y=\frac{\partial\tilde{u}}{\partial t}. \tag{3.3}$$

Then we have

$$\begin{cases} Py(x,t)=\dfrac{\partial}{\partial t}y(x,t)+\dfrac{\partial}{\partial x}y(x,t)+p(x)y(x,t)=f(x)\partial_t R(x,t), \\ y(x,0)=f(x)R(x,0), \quad 0<x<\ell, \\ y(0,t)=0, \quad -T<t<T, \end{cases} \tag{3.4}$$

where $\partial_t R$ means $\frac{\partial}{\partial t}R$. Thus, the initial condition is nonzero if $R(x,0)\neq 0$.

4 Carleman Estimate

Let us define

$$\varphi(x,t)=(x-x_0)^2-\beta t^2, \tag{4.1}$$

where $x_0 \notin [0,\ell]$ and $0<\beta<1$. We chose $\varphi(x,t)$ so that we have the following inequality.

Proposition 4.1. For the function $w(x,t) \in C_0^1([0,\ell]\times[-T,T])$ which satisfies $w(0,t)=w(x,\pm T)=0$, there exist $C>0$ and $s_0>0$ such that

$$\int_0^{\ell}\int_{-T}^{T} sw^2 \mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t \leq C\int_0^{\ell}\int_{-T}^{T}|Pw|^2\mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t + Cs\int_{-T}^{T} w^2(\ell,t)\mathrm{e}^{2s\varphi(\ell,t)}\mathrm{d}t \tag{4.2}$$

for all $s \geq s_0$.

Proof. We let P_0 denote the principal part of the transport equation.

$$P_0 w(x,t) = \partial_t w(x,t) + \partial_x w(x,t), \quad 0 < x < \ell, \quad -T < t < T. \tag{4.3}$$

We set $z = \mathrm{e}^{s\varphi} w$. Then we have

$$P_0 z = sA(x,t)z + \mathrm{e}^{s\varphi} P_0 w. \tag{4.4}$$

where we defined

$$A(x,t) = \partial_t \varphi + \partial_x \varphi = -2\beta t + 2(x - x_0). \tag{4.5}$$

Hence

$$\begin{aligned}
\int_0^\ell \int_{-T}^T |P_0 w|^2 \mathrm{e}^{2s\varphi} \mathrm{d}x\mathrm{d}t &= \int_0^\ell \int_{-T}^T (P_0 z - sAz)^2 \, \mathrm{d}x\mathrm{d}t \\
&\ge -2s \int_0^\ell \int_{-T}^T A(\partial_t z + \partial_x z) z \mathrm{d}x\mathrm{d}t \\
&= -s \int_0^\ell \int_{-T}^T A\left(\partial_t z^2 + \partial_x z^2\right) \mathrm{d}x\mathrm{d}t \\
&= s \int_0^\ell \int_{-T}^T (\partial_t A + \partial_x A) z^2 \mathrm{d}x\mathrm{d}t - s \int_{-T}^T A(\ell,t) z^2(\ell,t) \mathrm{d}t \\
&= 2s(1-\beta) \int_0^\ell \int_{-T}^T z^2 \mathrm{d}x\mathrm{d}t - s \int_{-T}^T A(\ell,t) z^2(\ell,t) \mathrm{d}S\mathrm{d}t \\
&\ge 2s(1-\beta) \int_0^\ell \int_{-T}^T z^2 \mathrm{d}x\mathrm{d}t - Cs \int_{-T}^T z^2(\ell,t) \mathrm{d}t. \qquad (4.6)
\end{aligned}$$

Thus we obtain

$$2(1-\beta) \int_0^\ell \int_{-T}^T s w^2 \mathrm{e}^{2s\varphi} \mathrm{d}x\mathrm{d}t \le \int_0^\ell \int_{-T}^T |P_0 w|^2 \mathrm{e}^{2s\varphi} \mathrm{d}x\mathrm{d}t + Cs \int_{-T}^T w^2(\ell,t) \mathrm{e}^{2s\varphi(\ell,t)} \mathrm{d}t. \tag{4.7}$$

Since $P_0 w = Pw - pw$, Eq. (4.2) is readily obtained from the above inequality. $\square$

5 Lipschitz Stability

If we assume $\partial_t R$ is sufficiently small, we can derive the Lipschitz stability by energy estimates (see Appendix A). In order not to have this (almost) unphysical condition, we need to use the Carleman estimate, which we derived in Sec. 4.

Let us choose T such that

$$T > \frac{1}{\sqrt{\beta}} \sup_{0 \le x \le \ell} |x - x_0|. \tag{5.1}$$

Note that $x_0 \notin [0, \ell]$. For this T, we have

$$\varphi(x, \pm T) = (x - x_0)^2 - \beta T^2 < 0, \quad x \in [0, \ell], \tag{5.2}$$

and

$$\varphi(x, 0) > 0, \quad x \in [0, \ell]. \tag{5.3}$$

Therefore, we can introduce small $\varepsilon > 0$, and choose small $\delta > 0$ such that

$$\varphi(x,t) < -\varepsilon, \quad -T \le t \le -T+2\delta,\ T-2\delta \le t \le T,\ x \in [0,\ell], \tag{5.4}$$

and

$$\varphi(x,t) > \varepsilon, \quad -\delta \le t \le \delta,\ x \in [0,\ell]. \tag{5.5}$$

Next we introduce a cut-off function $\mu \in C_0^\infty(\mathbb{R})$ which is $0 \le \mu \le 1$ and satisfies

$$\mu(t) = \begin{cases} 1, & -T+2\delta \le t \le T-2\delta, \\ 0, & -T \le t \le -T+\delta,\ T-\delta \le t \le T. \end{cases} \tag{5.6}$$

We set

$$w = \mu y. \tag{5.7}$$

Since $Py = f\partial_t R$, the function $w(x,t)$ satisfies

$$\partial_t w + \partial_x w + pw = \mu f \partial_t R + (\partial_t \mu) y, \qquad w(0,\cdot) = w(\cdot, \pm T) = 0. \tag{5.8}$$

Let us use Proposition 4.1. for the function w;

$$\begin{aligned} s\int_0^\ell \int_{-T}^T w^2 \mathrm{e}^{2s\varphi} \mathrm{d}x\mathrm{d}t \;&\le\; C\int_0^\ell f^2 \left(\int_{-T}^T \mathrm{e}^{-2s\beta t^2} \mathrm{d}t \right) \mathrm{e}^{2s\varphi(x,0)} \mathrm{d}x + C\int_0^\ell \int_{-T}^T (\partial_t\mu)^2 y^2 \mathrm{e}^{2s\varphi} \mathrm{d}x\mathrm{d}t \\ &+\; Cs\int_{-T}^T w^2(\ell,t) \mathrm{e}^{2s\varphi(\ell,t)} \mathrm{d}t. \end{aligned} \tag{5.9}$$

By the Lebesgue theorem[2], we have

$$\int_{-T}^T \mathrm{e}^{-2s\beta t^2} \mathrm{d}t = o(1) \quad \text{as } s \to \infty. \tag{5.10}$$

Here, o is the small O (the left-hand side is asymptotically dominated by 1). By using the fact that $\partial_t \mu \ne 0$ only for $-T+\delta \le t \le -T+2\delta$ or $T-2\delta \le t \le T-\delta$ and $\varphi(x,t) < -\varepsilon$ in these regions. We obtain

$$\int_0^\ell \int_{-T}^T (\partial_t\mu)^2 y^2 \mathrm{e}^{2s\varphi} \mathrm{d}x\mathrm{d}t \le C\mathrm{e}^{-2s\varepsilon} \int_0^\ell \int_{-T}^T y^2 \mathrm{d}x\mathrm{d}t. \tag{5.11}$$

Therefore we have

$$\begin{aligned} s\int_0^\ell \int_{-T}^T w^2 \mathrm{e}^{2s\varphi} \mathrm{d}x\mathrm{d}t \;&\le\; o(1)\int_0^\ell f^2 \mathrm{e}^{2s\varphi(x,0)} \mathrm{d}x + C\mathrm{e}^{-2s\varepsilon} \int_0^\ell \int_{-T}^T y^2 \mathrm{d}x\mathrm{d}t \\ &+\; C\mathrm{e}^{Cs} \int_{-T}^T y^2(\ell,t) \mathrm{d}t. \end{aligned} \tag{5.12}$$

[2] For functions $\eta_n \in L^1(-T,T)$ such that $\eta_n \to \eta$ (a.e.), and $\sup_n |\eta_n| \le M$, we have $\eta \in L^1(-T,T)$ and

$$\lim_{n\to\infty} \int_{-T}^T \eta_n \mathrm{d}x = \int_{-T}^T \eta \mathrm{d}x.$$

See Theorem I.16 in [11] or Theorem 1.8 in [12].

In the third term on the right-hand side of Eq. (5.12), we used $s\exp[2s\max_t \varphi(\ell,t)] \le \mathrm{e}^{Cs}$. We will use Eq. (5.12) later.

Let us introduce

$$z = w\mathrm{e}^{s\varphi} = \mu y \mathrm{e}^{s\varphi}. \tag{5.13}$$

The function $z(x,t)$ satisfies

$$\partial_t z + \partial_x z + pz = \mu \mathrm{e}^{s\varphi} f \partial_t R + (\partial_t \mu) y \mathrm{e}^{s\varphi} + s(\partial_t\varphi + \partial_x\varphi) z, \qquad z(0,\cdot) = z(\cdot, \pm T) = 0. \tag{5.14}$$

Let us multiply Eq. (5.14) by $-2z$ and integrate the equation over x and t.

$$\begin{aligned} -\int_0^\ell \int_0^T \partial_t z^2 \mathrm{d}x\mathrm{d}t &= \int_0^\ell \int_0^T \partial_x z^2 \mathrm{d}x\mathrm{d}t + \int_0^\ell \int_0^T 2pz^2 \mathrm{d}x\mathrm{d}t \\ &- \int_0^\ell \int_0^T 2\left\{\mu \mathrm{e}^{s\varphi} f(\partial_t R) z + (\partial_t\mu) y \mathrm{e}^{s\varphi} z + s(\partial_t\varphi + \partial_x\varphi) z^2\right\} \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{5.15}$$

Let us introduce $a_0 > 0$ such that $R(x,0) = a(x) \ge a_0$ in $[0,\ell]$. The left-hand side is calculated as

$$\mathrm{LHS} = \int_0^\ell z^2(x,0)\mathrm{d}x = \int_0^\ell y^2(x,0)\mathrm{e}^{2s\varphi(x,0)}\mathrm{d}x \ge a_0^2 \int_0^\ell f^2(x)\mathrm{e}^{2s\varphi(x,0)}\mathrm{d}x, \tag{5.16}$$

because $y(x,0) = f(x)R(x,0)$. The right-hand side is estimated as

$$\begin{aligned} \mathrm{RHS} &\le \int_0^T z^2(\ell,t)\mathrm{d}t + Cs\int_0^\ell \int_0^T z^2 \mathrm{d}x\mathrm{d}t \\ &+ \int_0^\ell \int_0^T (\partial_t\mu)^2 y^2 \mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t + \int_0^\ell \int_0^T f^2 \mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{5.17}$$

where we used the Cauchy-Bunyakovskii inequality[3]. Thus we have

$$\begin{aligned} a_0^2 \int_0^\ell f^2 \mathrm{e}^{2s\varphi(x,0)}\mathrm{d}x &\le \int_0^\ell \int_{-T}^T f^2 \mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t + \int_0^\ell \int_{-T}^T (\partial_t\mu)^2 y^2 \mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t \\ &+ Cs\int_0^\ell \int_{-T}^T w^2 \mathrm{e}^{2s\varphi}\mathrm{d}x\mathrm{d}t + \int_{-T}^T z^2(\ell,t)\mathrm{d}t. \end{aligned} \tag{5.18}$$

We use the inequality (5.12) in Eq. (5.18) and obtain

$$\begin{aligned} a_0^2 \int_0^\ell f^2 \mathrm{e}^{2s\varphi(x,0)}\mathrm{d}x &\le o(1)\int_0^\ell f^2 \mathrm{e}^{2s\varphi(x,0)}\mathrm{d}x + C\mathrm{e}^{-2s\varepsilon}\int_0^\ell \int_{-T}^T y^2 \mathrm{d}x\mathrm{d}t \\ &+ C\mathrm{e}^{Cs}\int_{-T}^T y^2(\ell,t)\mathrm{d}t. \end{aligned} \tag{5.19}$$

---

[3]

$$0 \le \int (f(x) + g(x))^2 \mathrm{d}x.$$

Therefore,

$$-\int 2f(x)g(x)\mathrm{d}x \le \int f^2(x)\mathrm{d}x + \int g^2(x)\mathrm{d}x.$$

We do similar calculation to prove the Cauchy-Bunyakovskii-Schwarz inequality, $\left(\int fg\mathrm{d}x\right)^2 \le \int f^2\mathrm{d}x + \int g^2\mathrm{d}x$.

Let us multiply $Py = f\partial_t R$ in Eq. (3.4) by $2y$ and integrate over x.

$$\int_0^\ell \partial_t y^2 \mathrm{d}x + \int_0^\ell \partial_x y^2 \mathrm{d}x + \int_0^\ell 2py^2 \mathrm{d}x = \int_0^\ell 2yf\partial_t R \mathrm{d}x. \tag{5.20}$$

The right-hand side may be estimated using the Cauchy-Bunyakovskii inequality. Thus,

$$\frac{\partial}{\partial t}\int_0^\ell y^2 \mathrm{d}x \le C\int_0^\ell y^2 \mathrm{d}x + C\int_0^\ell f^2 \mathrm{d}x, \tag{5.21}$$

where we used the fact that $y(0,t) = 0$ and

$$\int_0^\ell \partial_x y^2 \mathrm{d}x = y^2(\ell, t) \ge 0. \tag{5.22}$$

Then we integrate both sides over t.

$$\begin{aligned} \int_0^\ell y^2(x,t)\mathrm{d}x &\le \int_0^\ell y^2(x,0)\mathrm{d}x + C\int_0^t\int_0^\ell y^2 \mathrm{d}t\mathrm{d}x + C\int_0^t\int_0^\ell f^2 \mathrm{d}t\mathrm{d}x \\ &\le C||f||^2_{L^2(0,\ell)} + C\int_0^t\int_0^\ell y^2 \mathrm{d}t\mathrm{d}x. \end{aligned} \tag{5.23}$$

By using the Gronwall inequality[4], we obtain

$$\int_0^\ell y^2 \mathrm{d}x \le C||f||^2_{L^2(0,\ell)}. \tag{5.24}$$

Therefore,

$$\int_0^\ell \int_{-T}^T y^2 \mathrm{d}x\mathrm{d}t \le C||f||^2_{L^2(0,\ell)}, \quad -T < t < T. \tag{5.25}$$

Note that Eq. (5.5) implies

$$\varphi(x,0) = (x - x_0)^2 > \varepsilon + \beta\delta^2. \tag{5.26}$$

Using Eqs. (5.19), (5.25), and (5.26), we obtain

$$\left(a_0^2 - o(1)\right)\mathrm{e}^{2s(\varepsilon+\beta\delta^2)}\int_0^\ell f^2(x)\mathrm{d}x \le C\mathrm{e}^{-2s\varepsilon}||f||^2_{L^2(0,\ell)} + C\mathrm{e}^{Cs}\int_{-T}^T y^2(\ell,t)\mathrm{d}t. \tag{5.27}$$

Therefore, there exists s_0 such that

$$\left\{\left(a_0^2 - o(1)\right)\mathrm{e}^{2s(\varepsilon+\beta\delta^2)} - C\mathrm{e}^{-2s\varepsilon}\right\}||f||^2_{L^2(0,\ell)} \le C\mathrm{e}^{Cs}\int_{-T}^T y^2(\ell,t)\mathrm{d}t \tag{5.28}$$

for $s > s_0$. Thus we proved the following theorem.

[4]For

$$\eta(t) \le c_1 + c_2\int_0^t \eta(t_1)\mathrm{d}t_1,$$

we have

$$\eta(t) \le c_1 \mathrm{e}^{c_2 t}.$$

See Chapter III in [13].

Theorem 5.1 Let y satisfy Eq. (3.4) with R, $\partial_t R \in L^2(-T,T;L^\infty(0,\ell))$ and $p \in L^\infty(0,\ell)$. We assume $R(x,0) > 0$ in $[0,\ell]$ and choose $T > \sup_{x\in[0,\ell]} |x - x_0|$. Then, there exists a constant $C > 0$ such that

$$||f||_{L^2(0,\ell)} \leq C||y(\ell,\cdot)||_{L^2(-T,T)} \tag{5.29}$$

for any $f \in L^2(0,\ell)$.

6 Conclusion

By Theorem 6.1, we immediately obtain the following Lipschitz stability, which corresponds to Eq. (2.2)

Theorem 6.1 Let u satisfy Eq. (2.1) with $|a(x)| > 0$ in $[0,\ell]$. Let $T > \ell$. Let $||p||_{L^\infty(0,\ell)}$, $||q||_{L^\infty(0,\ell)} \leq M$ for a constant $M > 0$. Then, there exists a constant $C(\ell,T,a,b,M) > 0$ such that

$$||p - q||_{L^2(0,\ell)} \leq C \left|\left| \frac{\partial\left(u[p](\ell,\cdot) - u[q](\ell,\cdot)\right)}{\partial t} \right|\right|_{L^2(-T,T)}. \tag{6.1}$$

We conclude this lecture with several remarks.

Remark 6.2 Throughout this note, time is considered in $(-T,T)$. When we start at $t = 0$ and consider $t \in (0,T)$, we need to extend functions to $(-T,T)$. See [5].

Remark 6.3 In fact, our model (2.1) is almost too simple and we can solve it by the method of characteristics [9]. However, this is a nice toy model to illustrate the key idea of inverse problems by Carleman estimates. In general, it is very difficult to solve the transport equation because of the integral term due to scattering.

Remark 6.4 An application of this analysis is computed tomography (CT). In a CT scan, cancer in a human body is detected as inhomogeneity in $p(x)$ in Eq. (2.1). To obtain tomographic images, X-rays are sent in different directions and measured at different places. So, in addition to x and t in the intensity u, we need to take angle (the direction in which X-rays propagate) into account even though each X-ray penetrates without changing its direction as we assumed in this note. The analysis which we developed in this lecture gives an "intrinsic" stability of X-ray CT.

Remark 6.5 As we saw in Eq. (5.27), $R(x,0)$ [$= u[p](x,0)$ (see Eq. (3.1))] must be nonzero. We see that this rather strong condition seems to hold when we remember every day we are exposed to X-rays (e.g., X-rays from ^{40}K in a concrete wall of a building).

Remark 6.6 An alternative approach to inverse transport problems is to use the albedo operator $\mathcal{A}$, which is similar to the Dirichlet-to-Neumann map. The operator $\mathcal{A}$ maps $u(0,t)$ to $u(\ell,t)$. See a recent review [14] and references therein for this approach.

A Small Business

By using energy estimates, we can show the Lipschitz stability if $\partial_t R(x,t)$ is small.

Let split y into two parts:

$$y(x,t) = y_1(x,t) + y_2(x,t), \tag{A.1}$$

where y_1 and y_2 respectively satisfy

$$\begin{cases} Py_1(x,t) = f(x)\partial_t R(x,t), \\ y_1(x,0) = 0, \quad 0 < x < \ell, \\ y_1(0,t) = 0, \quad -T < t < T, \end{cases} \qquad \begin{cases} Py_2(x,t) = 0, \\ y_2(x,0) = f(x)R(x,0), \quad 0 < x < \ell, \\ y_2(0,t) = 0, \quad -T < t < T. \end{cases} \tag{A.2}$$

For y_1, we have

$$\int_0^\ell 2y_1 \left(\partial_t y_1 + \partial_x y_1 + py_1\right) \mathrm{d}x = \int_0^\ell 2y_1 f \partial_t R \mathrm{d}x. \tag{A.3}$$

We obtain

$$\frac{\partial}{\partial t} \int_0^\ell y_1^2 \mathrm{d}x \leq C \int_0^\ell y_1^2 \mathrm{d}x + \int_0^\ell \left(f\partial_t R\right)^2 \mathrm{d}x. \tag{A.4}$$

By using the Gronwall inequality, we obtain

$$||y_1||_{L^1(-T,T;L^2(0,\ell))} \leq C||f\partial_t R||_{L^1(-T,T;L^2(0,\ell))}. \tag{A.5}$$

For y_2, we have

$$\int_0^\ell 2y_2 \left(\partial_t y_2 + \partial_x y_2 + py_2\right) \mathrm{d}x = 0. \tag{A.6}$$

We obtain

$$||fR(\cdot,0)||_{L^2(0,\ell)} \leq C||y_2||_{L^1(-T,T;L^2(0,\ell))}. \tag{A.7}$$

Since $R(\cdot,0) \geq a_0 > 0$, we have

$$\begin{aligned} a_0||f||_{L^2(0,\ell)} &\leq ||fR(\cdot,0)||_{L^2(0,\ell)} \\ &\leq C||y - y_1||_{L^1(-T,T;L^2(0,\ell))} \\ &\leq C||y||_{L^1(-T,T;L^2(0,\ell))} + C||y_1||_{L^1(-T,T;L^2(0,\ell))} \\ &\leq C||y||_{L^1(-T,T;L^2(0,\ell))} + C||f||_{L^2(0,\ell)}||\partial_t R||_{L^1(-T,T;L^\infty(0,\ell))}. \end{aligned} \tag{A.8}$$

Therefore, we obtain

$$\left(a_0 - C||\partial_t R||_{L^1(-T,T;L^\infty(0,\ell))}\right) ||f||_{L^2(0,\ell)} \leq C||y||_{L^1(-T,T;L^2(0,\ell))}. \tag{A.9}$$

Theorem A.1. Let $\min_{x\in[0,\ell]} |R(x,0)| > 0$. For sufficiently small $||\partial_t R||_{L^1(-T,T;L^\infty(0,\ell))}$, we have

$$||f||_{L^2(0,\ell)} \leq C||y||_{L^1(-T,T;L^2(0,\ell))}. \tag{A.10}$$

We can similarly obtain the Lipschitz stability as Eq. (A.10) for other differential equations such as the wave equation. To remove the smallness condition for $\partial_t R$, we use Carleman estimates.

References

[1] Carleman, T., *Sur un problème d'unicité pour les systèmes d'équations aux dérivées partielles à deux variables indépendantes*, Ark. Mat. Astr. Fys. **26 B**, 1–9 (1939).

[2] Hörmander, L., *Linear Partial Differential Operators*, Springer-Verlag, Berlin (1963).

[3] Bukhgeĭm, A. L., Klibanov, M. V., *Uniqueness in the large of a class of multidimensional inverse problems* (Russian), Dokl. Akad. Nauk SSSR **260**, 269–272 (1981).

[4] Yamamoto, M., *Uniqueness and Stability in Multidimensional Hyperbolic Inverse Problems*, J. Math. Pures Appl. **78**, 65–98 (1999).

[5] Imanuvilov, O. Yu. and Yamamoto, M., *Global Uniqueness and Stability in Determining Coefficients of Wave Equations*, Comm. P. D. E. **26**, 1409–1425 (2001).

[6] Yamamoto, M., *Carleman estimates for parabolic equations and applications*, Inverse Problems **25**, 123013 (75pp) (2009).

[7] Isakov, V., *Inverse Problems for Partial Differential Equations*, Springer-Verlag, Berlin (1998).

[8] Special Lectures on Basic Mathematical Sciences V (901-46), Winter 2010, Department of Mathematical Sciences, The University of Tokyo.

[9] Case, K. M. and Zweifel, P. F., *Linear Transport Theory*, Addison-Wesley (1967).

[10] Ryzhik, L., Papanicolaou, and G., Keller, J. B., *Transport equations for elastic and other waves in random media*, Wave Motion **24**, 327–370 (1996).

[11] Reed, M. and Simon, B., *Methods of Modern Mathematical Physics I: Functional Analysis*, Academic Press (1972).

[12] Lieb, E. H. and Loss, M., *Analysis*, 2nd ed. American Mathematical Society (2001).

[13] Hartman, P., *Ordinary Differential Equations*, 2nd ed. SIAM (2002).

[14] Bal, G., *Inverse transport theory and applications*, Inverse Problems **25** 053001 (48pp) (2009).