Chapter 7
Time-dependent differential equations

Euler's method

First-order ordinary differential equations (ODE) are written as

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(x), \quad t > 0, \\ x(0) = x_0. \end{cases}$$

If $x(t)$ is the position of a particle moving on the x-axis at time t, then $\mathrm{d}x/\mathrm{d}t$ is the velocity of the particle, which depends on the position in general. The initial position of the particle is x_0.

Example 1. Let us look at a few examples.

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= x, \quad x(0) = 1 \qquad \Rightarrow \qquad x(t) = \mathrm{e}^t, \\ \frac{\mathrm{d}x}{\mathrm{d}t} &= x^2, \quad x(0) = 1 \qquad \Rightarrow \qquad x(t) = \frac{1}{1-t}, \\ \frac{\mathrm{d}x}{\mathrm{d}t} &= \sin x, \quad x(0) = 1 \qquad \Rightarrow \qquad x(t) = ?. \end{aligned}$$

The solution to the last equation is not obvious[1].

Sometimes it is enough to obtain numerical solutions. Moreover in many cases, only numerical solutions are available.

The simplest numerical method is Euler's method.

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Introduction to Numerical Methods

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[1] The last example is not too difficult. We define $y = \tan(x/2)$. We have $1 + y^2 = \left[\cos^2(x/2)\right]^{-1}$, $\mathrm{d}y/\mathrm{d}x = (1 + y^2)/2$, and $\sin(x) = 2\sin(x/2)\cos(x/2) = 2\cos^2(x/2)\tan(x/2) = 2y/\left[\cos^2(x/2)\right]^{-1} = 2y/(1 + y^2)$. Hence we obtain $\mathrm{d}y/\mathrm{d}t = (\mathrm{d}y/\mathrm{d}x)(\mathrm{d}x/\mathrm{d}t) = (1/2)(1 + y^2)\sin(x) = y$. We obtain $y = C\mathrm{e}^t$ with a constant C. The initial condition $x(0) = 1$ implies $y(0) = \tan(x(0)/2) = \tan(1/2) = C$. Finally we obtain $x(t) = 2\tan^{-1}\left(\mathrm{e}^t \tan(1/2)\right)$.

We choose time step Δt and discretize the time t as $t_i = i\Delta t$ $(i = 0, 1, \ldots n)$. In Euler's method, numerical solution $w_i \approx x(t_i)$ is obtained as

$$\frac{w_{i+1} - w_i}{\Delta t} = f(w_i) \qquad \Longleftrightarrow \qquad w_{i+1} = w_i + \Delta t\, f(w_i).$$

Starting from $w_0 = x_0$, we can compute $w_1, w_2, \ldots$.

The error for Euler's method is given as

$$|x(t_i) - w_i| \leq C\Delta t,$$

where $C > 0$ is a constant.

Runge-Kutta methods

Second order

Euler's method is an $O(\Delta t)$ approximation. Let us consider higher order approximations.

We consider

$$\frac{w_{i+1} - w_i}{\Delta t} = \phi(w_i) = a_1 f(w_i) + a_2 f(w_i + \delta f(w_i)). \tag{7.1}$$

We want to choose a_1, a_2, δ so that the right-hand side provides an $O\left((\Delta t)^2\right)$ approximation. This method is called the second-order Runge-Kutta method.

Let us look at

$$\frac{x(t_{i+1}) - x(t_i)}{\Delta t} = \phi(x(t_i)) + \tau_i, \tag{7.2}$$

where τ_i is the error. Suppose

$$|\tau_i| \leq M,$$

for all $i = 0, 1, \ldots, n$. We can show that

$$|x(t_i) - w_i| \leq \frac{\mathrm{e}^{t_i L} - 1}{L} M, \tag{7.3}$$

where L is a constant such that $|\phi(x(t_i)) - \phi(w_i)| \leq L|x(t_i) - w_i|$ (we assume that ϕ satisfies the Lipschitz condition). This is proved as follows. From (7.1) we have $w_{i+1} = w_i + \Delta t\, \phi(w_i)$. From (7.2) we have $x(t_{i+1}) = x(t_i) + \Delta t \phi(x(t_i)) + \Delta t \tau_i$. By subtraction we obtain

$$x(t_{i+1}) - w_{i+1} = x(t_i) - w_i + \Delta t\, [\phi(x(t_i)) - \phi(w_i)] + \Delta t \tau_i.$$

Thus,

$$|x(t_{i+1}) - w_{i+1}| \le |x(t_i) - w_i| + \Delta t\,|\phi(x(t_i)) - \phi(w_i)| + \Delta t|\tau_i| \le a|x(t_i) - w_i| + b,$$

where $a = 1 + \Delta tL$, $b = \Delta tM$. We obtain

$$\begin{aligned}
|x(t_{i+1}) - w_{i+1}| &\le a|x(t_i) - w_i| + b \le a\,(a|x(t_{i-1}) - w_{i-1}| + b) + b \le \cdots \\
&\le a^{i+1}|x(t_0) - w_0| + (1 + a + a^2 + \cdots + a^i)b = a^{i+1}|x(t_0) - w_0| + \frac{a^{i+1}-1}{a-1}b \\
&= a^{i+1}\left(|x_0 - w_0| + \frac{b}{a-1}\right) - \frac{b}{a-1} \\
&= \left[(1+\Delta tL)^{i+1} - 1\right]\frac{\Delta tM}{\Delta tL}.
\end{aligned}$$

We note that $1 + \Delta tL \le \mathrm{e}^{\Delta tL}$ (use Taylor series) and $(i+1)\Delta t = t_{i+1}$. We obtain

$$|x(t_{i+1}) - w_{i+1}| \le \left(\mathrm{e}^{t_{i+1}L} - 1\right)\frac{M}{L}.$$

Thus (7.3) was proved. From (7.3), we see that an $O\left((\Delta t)^2\right)$ approximation is obtained if $M = O\left((\Delta t)^2\right)$.

Let us consider (7.2). We will choose a_1, a_2, δ so that $\tau_i = O\left((\Delta t)^2\right)$. The Taylor series of $x(t)$ about t_i is written as

$$x(t) = x(t_i) + \left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{t_i}(t - t_i) + \frac{1}{2}\left.\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right|_{t_i}(t - t_i)^2 + \cdots.$$

Evaluating the resulting expression at $t = t_{i+1}$, we obtain

$$\begin{aligned}
\frac{x(t_{i+1}) - x(t_i)}{\Delta t} &= \frac{\mathrm{d}x}{\mathrm{d}t}(t_i) + \frac{1}{2}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}(t_i)\Delta t + \cdots \\
&= f(x(t_i)) + \frac{1}{2}\Delta t\frac{\mathrm{d}f}{\mathrm{d}x}(x(t_i))f(x(t_i)) + \cdots.
\end{aligned} \tag{7.4}$$

By setting $\Delta x = \delta f(x(t_i))$, we obtain

$$\begin{aligned}
\phi(x(t_i)) &= a_1 f(x(t_i)) + a_2 f(x(t_i) + \Delta x) \\
&= a_1 f(x(t_i)) + a_2\left[f(x(t_i)) + \Delta x\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x(t_i)} + \frac{(\Delta x)^2}{2}\left.\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right|_{x(t_i)} + \cdots\right] \\
&= (a_1 + a_2)f(x(t_i)) + a_2\delta\frac{\mathrm{d}f}{\mathrm{d}x}(x(t_i))f(x(t_i)) \\
&+ \frac{a_2}{2}\delta^2\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x(t_i))f(x(t_i))^2 + \cdots.
\end{aligned} \tag{7.5}$$

By using (7.4) and (7.5), we obtain τ_i in (7.2) as

$$\tau_i = \frac{x(t_{i+1}) - x(t_i)}{\Delta t} - \phi(x(t_i))$$
$$= f(x(t_i)) + \frac{1}{2}\Delta t \frac{\mathrm{d}f}{\mathrm{d}x}(x(t_i))f(x(t_i)) - (a_1 + a_2)f(x(t_i)) - a_2\delta\frac{\mathrm{d}f}{\mathrm{d}x}(x(t_i))f(x(t_i))$$
$$- \frac{a_2}{2}\delta^2\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x(t_i))f(x(t_i))^2 + \cdots.$$

Therefore by choosing a_1, a_2, δ such that

$$a_1 + a_2 = 1, \qquad a_2\delta = \frac{\Delta t}{2},$$

we have $\tau_i = O\left((\Delta t)^2\right)$ and the method becomes an $O\left((\Delta t)^2\right)$ approximation:

$$|x(t_i) - w_i| = O\left((\Delta t)^2\right).$$

Although there are infinitely many second-order Runge-Kutta methods, there are a few common choices of a_1, a_2, δ

The modified Euler method

Let us set

$$a_1 = 0, \quad a_2 = 1, \quad \delta = \frac{\Delta t}{2}.$$

We have

$$\frac{w_{i+1} - w_i}{\Delta t} = f\left(w_i + \frac{\Delta t}{2}f(w_i)\right).$$

The formula is understood as follows. By the midpoint integration we have

$$x(t_{i+1}) - x(t_i) = \int_{t_i}^{t_{i+1}} x'\mathrm{d}t \approx (t_{i+1} - t_i)x'\left(\frac{t_i + t_{i+1}}{2}\right) = \Delta t\, x'\left(t_i + \frac{\Delta t}{2}\right).$$

By Euler's method with step $\Delta t/2$ we have

$$x\left(t_i + \frac{\Delta t}{2}\right) \approx x(t_i) + \frac{\Delta t}{2}f(x(t_i)).$$

Therefore,

$$x(t_{i+1}) - x(t_i) \approx \Delta t\, f\left(x(t_i) + \frac{\Delta t}{2}f(x(t_i))\right).$$

The algorithm is summarized as follow.

With the modified Euler method, we obtain $w_i \approx x(t_i)$ by the following two steps.

- $\tilde{w} = w_i + \dfrac{\Delta t}{2} f(w_i)$ Euler's method with step $\Delta t/2$
- $w_{i+1} = w_i + \Delta t\, f(\tilde{w})$ Midpoint integration

Heun's method

Here we set
$$a_1 = a_2 = \frac{1}{2}, \quad \delta = \Delta t.$$
We have
$$\frac{w_{i+1} - w_i}{\Delta t} = \frac{1}{2}\left[f(w_i) + f\left(w_i + \Delta t f(w_i)\right)\right].$$
The formula is understood as follows. By the trapezoidal integration we have
$$x(t_{i+1}) - x(t_i) = \int_{t_i}^{t_{i+1}} x' \mathrm{d}t \approx \frac{t_{i+1} - t_i}{2}\left[x'(t_i) + x'(t_{i+1})\right] = \frac{\Delta t}{2}\left[f(x(t_i)) + f(x(t_{i+1}))\right].$$
By Euler's method with step Δt we have
$$x(t_{i+1}) \approx x(t_i) + \Delta t f(x(t_i)).$$
Therefore,
$$x(t_{i+1}) - x(t_i) \approx \frac{\Delta t}{2}\left[f(x(t_i)) + f(x(t_i) + \Delta t f(x(t_i))\right].$$
This is also an improved Euler's method

The algorithm is summarized as follow.

With the Heun method, we obtain $w_i \approx x(t_i)$ as
$$w_{i+1} = w_i + \frac{1}{2}(k_1 + k_2),$$
where
$$k_1 = \Delta t f(w_i), \qquad k_2 = \Delta t f(w_i + k_1).$$

Fourth order

We can also construct the fourth-order Runge-Kutta method. The scheme is summarized as follows.

The fourth-order Runge-Kutta method updates the approximate solution at each time step according to the formula

$$w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

where

$$k_1 = \Delta t f(w_i), \quad k_2 = \Delta t f\left(w_i + \frac{k_1}{2}\right), \quad k_3 = \Delta t f\left(w_i + \frac{k_2}{2}\right), \quad k_4 = \Delta t f(w_i + k_3).$$

The error is $O\left((\Delta t)^4\right)$:

$$|x(t_i) - w_i| \leq C(\Delta t)^4,$$

where C is a positive constant.

The procedure is explained as follows. We begin with

$$x(t_{i+1}) - x(t_i) = \int_{t_i}^{t_{i+1}} \frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t = \int_{t_i}^{t_i+\Delta t} x'(t)\mathrm{d}t.$$

By Simpson's rule, we have

$$\begin{aligned} x(t_{i+1}) - x(t_i) &\approx \frac{\Delta t}{6}\left[x'(t_i) + 4x'\left(t_i + \frac{\Delta t}{2}\right) + x'(t_{i+1})\right] \\ &= \frac{\Delta t}{6}\left[x'(t_i) + 2x'\left(t_i + \frac{\Delta t}{2}\right) + 2x'\left(t_i + \frac{\Delta t}{2}\right) + x'(t_{i+1})\right]. \end{aligned}$$

We replace the four terms on the right-hand side with k_1, k_2, k_3, k_4 as follows.

$$\Delta t\, x'(t_i) = \Delta t\, f(x(t_i)) \approx \Delta t\, f(w_i) = k_1,$$

$$\begin{aligned} \Delta t\, x'\left(t_i + \frac{\Delta t}{2}\right) &= \Delta t\, f\left(x(t_i + \frac{\Delta t}{2})\right) = \Delta t\, f\left(x(t_i) + \int_{t_i}^{t_i+\frac{\Delta t}{2}} x'(t)\mathrm{d}t\right) \\ &\approx \Delta t\, f\left(x(t_i) + \frac{\Delta t}{2}x'(t_i)\right) \approx \Delta t\, f\left(w_i + \frac{k_1}{2}\right) = k_2. \end{aligned}$$

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\begin{aligned}
\Delta t\, x'\left(t_i + \frac{\Delta t}{2}\right) &= \Delta t\, f\left(x(t_i + \frac{\Delta t}{2})\right) = \Delta t\, f\left(x(t_i) + \int_{t_i}^{t_i + \frac{\Delta t}{2}} x'(t)\mathrm{d}t\right) \\
&\approx \Delta t\, f\left(x(t_i) + \frac{\Delta t}{2} x'(t_i + \frac{\Delta t}{2})\right) \approx \Delta t\, f\left(w_i + \frac{k_2}{2}\right) = k_3.
\end{aligned}
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\begin{aligned}
\Delta t\, x'(t_{i+1}) &= \Delta t\, f\left(x(t_{i+1})\right) = \Delta t\, f\left(x(t_i) + \int_{t_i}^{t_i + \Delta t} x'(t)\mathrm{d}t\right) \\
&\approx \Delta t\, f\left(x(t_i) + \Delta t\, x'(t_i + \frac{\Delta t}{2})\right) \approx \Delta t\, f(w_i + k_3) = k_4.
\end{aligned}
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