Problem Set 6 (10/28, 30)
Due on Fri, Nov 15

1) Consider matrix $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$.

(a) Find the eigenvalues λ_1, λ_2. Show your work.
(b) Find the orthonormal eigenvectors $\mathbf{q}_1, \mathbf{q}_2$. Show your work.
(c) Compute the Rayleigh quotient $R_A(\mathbf{x})$ for $\mathbf{x} = \mathbf{q}_1, \mathbf{q}_2$. Show your work.
(d) Is $R_A(\mathbf{q}_1 + \mathbf{q}_2) = \lambda_1 + \lambda_2$? Explain.

2) Let us obtain the largest and smallest eigenvalues λ_1, λ_3 of A by the power method and inverse power method starting from the initial guess $\mathbf{x}^{(0)}$:

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix}, \quad \mathbf{x}^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

(a) Find the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ ($|\lambda_1| > |\lambda_2| > |\lambda_3|$) using the `eig` command in MATLAB.
(b) Write a MATLAB code for the power method.
(c) Present the results for λ_1 in a table as follows. Take six steps, $k = 0, 1, \ldots, 5$. In columns 3 and 4, use $\lambda = \lambda_1$ which is obtained in part (a).

column 1: k $\leftarrow$ step index
column 2: $\lambda^{(k)}$ $\leftarrow$ approximate eigenvalue at step k
column 3: $|\lambda^{(k)} - \lambda|$ $\leftarrow$ error at step k
column 4: $\frac{|\lambda^{k} - \lambda|}{|\lambda^{(k-1)} - \lambda|}$ $\leftarrow$ ratio of errors at steps k and $k-1$

(d) Is the convergence factor for the power method $(\lambda_2/\lambda_1)^2$? Explain.
(e) Write a MATLAB code for the inverse power method.
(f) Present the results for λ_3 in a table following part (c). In columns 3 and 4, use $\lambda = \lambda_3$ which is obtained in part (a).
(g) Is the convergence factor for the inverse power method $(\lambda_3/\lambda_2)^2$? Explain.

3) Repeat 2) (a),(c),(d),(f) for the matrix A and vector $\mathbf{x}^{(0)}$ below.

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad \mathbf{x}^{(0)} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$