MATH 471 SECTION 002
MIDTERM

October 25, 2013,   Instructor: Manabu Machida

Name: ____

- To receive full credit you must show all your work.
- One side of a US letter size paper ($8.5'' \times 11''$) with notes is OK.
- You can use the back side of a paper if you need. Indicate where your calculation jumps.
- **NO CALCULATOR, SMARTPHONE, BOOKS, or OTHER NOTES.**

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
TOTAL	40	

Problem 1. (10 points) We consider two matrices $A_1 = \begin{pmatrix} 10 & 9 \\ 1 & 8 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 10 & 9 \\ 9 & 8 \end{pmatrix}$.

Let $\mathbf{b} = \begin{pmatrix} 10 \\ 9 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

(a) Solve $A_1\mathbf{x} = \mathbf{b}$ with the Gauss-Seidel method by rewriting the problem as $(L+D)\mathbf{x} = -U\mathbf{x} + \mathbf{b}$. Take two steps and find $x_1^{(k)}, x_2^{(k)}$ $(k = 1, 2)$ starting from initial guess $x_1^{(0)} = x_2^{(0)} = 0$.

(b) The matrix A_2 can be written as $A_2 = \begin{pmatrix} 1 & 0 \\ 0.9 & 1 \end{pmatrix} \begin{pmatrix} 10 & 9 \\ 0 & -0.1 \end{pmatrix}$. Solve $A_2\mathbf{x} = \mathbf{b}$ by using forward substitution and back substitution.

(c) Which is larger $\kappa_\infty(A_1)$ or $\kappa_\infty(A_2)$? (*Hint:* the condition number $\kappa(A) = \|A\|\,\|A^{-1}\|$).

Solution (a)

$$\begin{cases} 10x_1^{(k+1)} = -9x_2^{(k)} + 10, \\ 8x_2^{(k+1)} = -x_1^{(k+1)} + 9. \end{cases}$$

We obtain

$$x_1^{(1)} = \frac{10}{10} = 1, \qquad x_2^{(1)} = \frac{1}{8}\left(9 - x_1^{(1)}\right) = 1.$$

and

$$x_1^{(2)} = \frac{1}{10}\left(10 - 9x_2^{(1)}\right) = \frac{1}{10}, \qquad x_2^{(2)} = \frac{1}{8}\left(9 - x_1^{(2)}\right) = \frac{89}{80}.$$

(b) We have $A = LU$, where

$$L = \begin{pmatrix} 1 & 0 \\ 0.9 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 10 & 9 \\ 0 & -0.1 \end{pmatrix}.$$

The equation $L\mathbf{y} = \mathbf{b}$ is solved by forward substitution as

$$y_1 = 10, \qquad y_2 = 0.$$

Finally the equation $U\mathbf{x} = \mathbf{y}$ is solved by back substitution as

$$x_1 = 1, \qquad x_2 = 0.$$

(c) Note that

$$A_1^{-1} = \frac{1}{71}\begin{pmatrix} 8 & -9 \\ -1 & 10 \end{pmatrix}, \qquad A_2^{-1} = \begin{pmatrix} -8 & 9 \\ 9 & -10 \end{pmatrix}.$$

Therefore,

$$\kappa_\infty(A_1) = 19 \cdot \frac{17}{71} = \frac{323}{71} = 4.549\ldots, \qquad \kappa_\infty(A_2) = 19 \cdot 19 = 361.$$

Thus $\kappa_\infty(A_2)$ is larger than $\kappa_\infty(A_2)$.

(continued)

Problem 2. (10 points)

(a) Convert $(10.25)_{10}$ to base 2.

(b) Convert $(11.01)_2$ to base 10.

Solution (a) $(1010.01)_2$, (b) 3.25.

Problem 3. (10 points) Consider $f(x) = x^2 - 7$, which has one positive root r. Compute an approximation to r by Newton's method with the initial guess $x_0 = 3$. What is the first step solution x_1?

Solution

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0^2 - 7}{2x_0} = 3 - \frac{2}{6} = \frac{8}{3}.$$

Problem 4. (10 points) Consider $\begin{pmatrix} 10^{-10} & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1+10^{-10} \\ 2 \end{pmatrix}$.

(a) Find x_1, x_2 by Gaussian elimination (without pivoting).

(b) Suppose we solve the problem by Gaussian elimination in part (a) with a computer which has 8 significant digits (for example, 0.123456789×10^3 is rounded to 0.12345679×10^3). What solution would the computer return?

(c) Improve the algorithm in part (b). That is, find x_1, x_2 by Gaussian elimination with partial pivoting. Still use 8 significant digits. Show intermediate steps.

Solution (a)

$$\left(\begin{array}{cc|c} 10^{-10} & 1 & 1+10^{-10} \\ 1 & 1 & 2 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 10^{-10} & 1 & 1+10^{-10} \\ 0 & 1-10^{10} & 1-10^{10} \end{array}\right).$$

Therefore,

$$x_1 = 1, \qquad x_2 = 1.$$

(b)

$$\left(\begin{array}{cc|c} 10^{-10} & 1 & 1+10^{-10} \\ 1 & 1 & 2 \end{array}\right) = \left(\begin{array}{cc|c} 10^{-10} & 1 & 1 \\ 1 & 1 & 2 \end{array}\right)$$
$$\rightarrow \left(\begin{array}{cc|c} 10^{-10} & 1 & 1 \\ 0 & 1-10^{10} & 2-10^{10} \end{array}\right) = \left(\begin{array}{cc|c} 10^{-10} & 1 & 1 \\ 0 & -10^{10} & -10^{10} \end{array}\right).$$

Thus,

$$x_1 = 0, \qquad x_2 = 1.$$

(c)

$$\left(\begin{array}{cc|c} 10^{-10} & 1 & 1+10^{-10} \\ 1 & 1 & 2 \end{array}\right)$$
$$\rightarrow \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 10^{-10} & 1 & 1+10^{-10} \end{array}\right) = \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 10^{-10} & 1 & 1 \end{array}\right)$$
$$\rightarrow \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right).$$

Hence,

$$x_1 = 1, \qquad x_2 = 1.$$

(continued)