MATH 471 SECTION 002
FINAL

December 19, 2013, Instructor: Manabu Machida

Name: ____

- To receive full credit you must show all your work.
- A US letter size paper (8.5" × 11") with notes on both sides is OK.
- You can use the back side of a paper if you need. Indicate where your calculation jumps.
- **NO CALCULATOR, SMARTPHONE, BOOKS, or OTHER NOTES.**

Problem	Points	Score
1	10	
2	15	
3	10	
4	5	
5	15	
6	10	
7	5	
8	10	
TOTAL	80	

Problem 1. (10 points) Consider $f(x) = \cos x$ with the three points $x_0 = -\frac{\pi}{2}$, $x_1 = 0$, $x_2 = \frac{\pi}{2}$. Find Newton's form for the interpolating polynomial $p_2(x)$.

Solution We obtain

$$f[x_0] = f(x_0) = 0, \quad f[x_1] = f(x_1) = 1, \quad f[x_2] = f(x_2) = 0,$$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{2}{\pi}, \quad f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1} = -\frac{2}{\pi},$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = -\frac{4}{\pi^2}.$$

Therefore,

$$\begin{aligned} p_2(x) &= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\ &= \frac{2}{\pi}\left(x + \frac{\pi}{2}\right) - \frac{4}{\pi^2}\left(x + \frac{\pi}{2}\right) x \\ &= 1 - \frac{4}{\pi^2} x^2. \end{aligned}$$

Problem 2. (15 points)

(a) Find the diagonal matrix D below.

$$AS = SD, \quad A = \begin{pmatrix} 5 & 1 & -2 \\ 1 & 6 & -1 \\ -2 & -1 & 5 \end{pmatrix}, \quad S = \begin{pmatrix} -\sqrt{2} & -1 & \sqrt{3} \\ -\sqrt{2} & 2 & 0 \\ \sqrt{2} & 1 & \sqrt{3} \end{pmatrix}.$$

(b) The table below shows numerical results of finding the dominant eigenvalue λ_1 of A by power method with initial vector $\mathbf{x}_0^T = (1.2, -6, 12)$. What is the asymptotic rate a?

k	$\lambda^{(k)}$	$\lvert\lambda^{(k)} - \lambda_1\rvert$	$\lvert\lambda^{(k)} - \lambda_1\rvert / \lvert\lambda^{(k-1)} - \lambda_1\rvert$
0	1.015200e+03	1.007200e+03	–
1	7.421776e+00	5.782241e-01	5.740906e-04
2	7.908973e+00	9.102748e-02	1.574259e-01
3	7.986704e+00	1.329581e-02	1.460637e-01
			$\downarrow$
			a

Solution (a) We can write

$$AS = S\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} -8\sqrt{2} & * & * \\ * & 10 & * \\ * & * & 3\sqrt{3} \end{pmatrix} = \begin{pmatrix} -\sqrt{2}\lambda_1 & * & * \\ * & 2\lambda_2 & * \\ * & * & \sqrt{3}\lambda_3 \end{pmatrix}.$$

By comparing both sides, we find

$$D = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

(b) $\lambda_1 = 8 > \lambda_2 = 5 > \lambda_3 = 3 \quad \Rightarrow \quad a = (5/8)^2 \; (= 0.390625).$

(continued)

Problem 3. (10 points) Consider $x'(t) = f(x)$, $x(0) = 1$. Let us calculate numerical solution $w_i \approx x(t_i)$, where $t_i = i\Delta t$ $(i = 0, 1, \ldots)$.

(a) For $f(x) = x$, $\Delta t = 0.1$, compute w_1 by Euler's method.

(b) Let us improve Euler's method as follows. What are RHS1 and RHS2? Use $\Delta t, t_i, x'$ for RHS1, and use $\Delta t, t_i, x, f$ for RHS2.

By the midpoint integration we have

$$x(t_{i+1}) - x(t_i) = \int_{t_i}^{t_{i+1}} x' dt \approx \text{RHS1}.$$

By Euler's method with step $\Delta t/2$ we have

$$x\left(t_i + \frac{\Delta t}{2}\right) \approx \text{RHS2}.$$

(c) For $f(x) = x$, $\Delta t = 0.1$, compute w_1 by this improved Euler's method.

Solution (a) $w_1 = w_0 + \Delta t\, f(w_0) = 1 + 0.1 \cdot 1 = 1.1.$

(b)

$$\text{RHS1} = (t_{i+1} - t_i)\, x'\left(\frac{t_i + t_{i+1}}{2}\right) = \Delta t\, x'\left(t_i + \frac{\Delta t}{2}\right).$$

$$\text{RHS2} = x(t_i) + \frac{\Delta t}{2} f(x(t_i)).$$

(c) The improved version is obtained as

$$w_{i+1} = w_i + k, \quad k = \Delta t f\left(w_i + \frac{\Delta t}{2} f(w_i)\right).$$

Hence, $k = 0.1 \cdot (1 + 0.05 \cdot 1) = 0.105$, $w_1 = w_0 + k = 1 + 0.105 = 1.105$.

(continued)

Problem 4. (5 points) The positive root of $f(x) = x^2 - 3$ is $\sqrt{3} = 1.73205\ldots$. Let us calculate the root by the bisection method, fixed-point iteration, and Newton's method. The results are presented below. What are Methods A, B, and C?

n	Method A	Method B	Method C
0	1.5	1.5	1.5
1	2	1.75	1.75
2	1.5	1.732	1.625
3	2	1.73205	1.6875

Solution Method A = fixed-iteration ($x = 3/x$), Method B = Newton's method, and Method C = bisection.

Problem 5. (15 points) In the code below, elements of an upper triangular matrix A are stored in the array `a(:,:)`, and components of a vector $\mathbf{b}$ are stored in the array `b(:)`.

```
x(n)=b(n)/a(n,n)
for i=n-1:-1:1
  tmp=b(i)
  for j=i+1:n
    tmp=tmp-a(i,j)*x(j)
  end
  x(i)=tmp/a(1,1)
end
```

(a) What is the purpose of the code?

(b) Find and correct any bugs.

(c) Obtain the operation count for this algorithm.

Solution (a) The code obtains $\mathbf{x}$ in $A\mathbf{x} = \mathbf{b}$ by back substitution. (b) `x(i)=tmp/a(1,1)` should be `x(i)=tmp/a(i,i)`. (c) The number of divisions is n. The numbers of multiplications and additions are $1 + 2 + \cdots + (n-1) = \frac{1}{2}n(n-1)$, respectively. Therefore, the operation count is $n + 2 \cdot \frac{1}{2}n(n-1) = n^2$.

(continued)

Problem 6. (10 points) Consider the floating point representation $x = \pm(0.d_1 d_2 \cdots d_n)_\beta \cdot \beta^e$, $-M \le e \le M$. Suppose $\beta = n = M = 4$.

(a) How many different numbers can be represented?

(b) Find fl(33.75), the floating point representation of 33.75 in this system.

Solution (a) $2 \cdot 3 \cdot 4^3 \cdot 9 + 1 = 3457$. (b) $33.75 = 2 \cdot 4^2 + 1 \cdot 4^0 + 3 \cdot 4^{-1} = (201.3)_4 = (0.2013)_4 \cdot 4^3$.

Problem 7. (5 points) For a given function $f(x)$ on $[-1, 1]$, the cubic spline for uniform points $x_i = -1 + ih$, $h = 2/n$ $(i = 0, 1, \ldots, n)$ is obtained as follows. What is RHS below?

Step 1: We write $s_i''(x)$ as follows using a_i, a_{i+1}.

$$s_i''(x) = \text{RHS}, \qquad i = 0, 1, \ldots, n-1.$$

Then $s_i''(x_i) = a_i$, $s_i''(x_{i+1}) = a_{i+1}$, and $s_{i-1}''(x_i) = a_i = s_i''(x_i)$.

Step 2: We integrate $s_i''(x)$ twice. Note that $s_i(x_i) = f_i$.

Step 3: We differentiate $s_i(x)$. Note that $s_{i-1}'(x_i) = s_i'(x_i)$ $(i = 1, \ldots, n-1)$.

Step 4: Set $s''(x_0) = s''(x_n) = 0$. We obtain a_i by solving a matrix-vector equation.

Solution

$$\text{RHS} = a_i \left(\frac{x_{i+1} - x}{h} \right) + a_{i+1} \left(\frac{x - x_i}{h} \right).$$

Problem 8. (10 points) Consider $\int_0^1 x^2 dx$. We take $x_i = ih$, $h = 1/n$, $i = 0, 1, \ldots, n$.

(a) Find the trapezoid rule $T(h)$ for $h = 1$, 0.5.

(b) Find a better approximation by Richardson extrapolation.

Solution (a)

$$\begin{aligned} T(1) &= \left(\frac{1}{2}0^2 + \frac{1}{2}1^2\right) \cdot 1 = \frac{1}{2} = 0.5, \\ T(0.5) &= \left(\frac{1}{2}0^2 + 0.5^2 + \frac{1}{2}1^2\right) \cdot 0.5 = \frac{3}{8} = 0.375. \end{aligned}$$

(b)

$$R_1(0.5) = \frac{1}{3}\Big(4T(0.5) - T(2 \cdot 0.5)\Big) = \frac{1}{3}\left(4 \cdot \frac{3}{8} - \frac{1}{2}\right) = \frac{1}{3}.$$