MATH 454 SECTION 002

Quiz 5

February 21, 2014, Instructor: Manabu Machida

Name: ____

- To receive full credit you must show all your work.
- You can use the back side of a paper if you need. Indicate where your calculation jumps.
- **NO CALCULATOR, SMARTPHONE, BOOKS, or OTHER NOTES.**

Find the eigenvalues and eigenfunctions for the Sturm-Liouville eigenvalue problem

$$\phi''(x) + \lambda\phi(x) = 0, \quad \phi(0) = \phi(L), \quad \phi'(0) = \phi'(L).$$

[*Hint:* You can use the facts that $\phi(x) = 0$ if $\lambda < 0$, $\phi(x) = C$ (constant) if $\lambda = 0$, and we can write $\phi(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$ if $\lambda > 0$.]

Solution [6] Let us assume $\lambda > 0$. From the boundary conditions we obtain [2]

$$\begin{cases} \phi(0) = \phi(L), \\ \phi'(0) = \phi'(L), \end{cases} \quad \Rightarrow \quad \begin{cases} A = A\cos(\sqrt{\lambda}L) + B\sin(\sqrt{\lambda}L), \\ B = -A\sin(\sqrt{\lambda}L) + B\cos(\sqrt{\lambda}L). \end{cases} \tag{1}$$

We will consider two cases [1]. If $\sin(\sqrt{\lambda}L) \neq 0$, then

$$B = \frac{1 - \cos(\sqrt{\lambda}L)}{\sin(\sqrt{\lambda}L)} A.$$

Since we look for nontrivial solutions, we suppose $A \neq 0$. We obtain

$$[1 - \cos(\sqrt{\lambda}L)]B = -A\sin(\sqrt{\lambda}L) \quad \Rightarrow \quad [1 - \cos(\sqrt{\lambda}L)]^2 + \sin^2(\sqrt{\lambda}L) = 2[1 - \cos(\sqrt{\lambda}L)] = 0.$$

Thus, [1]

$$\cos(\sqrt{\lambda}L) = 1 \quad \Leftrightarrow \quad \sqrt{\lambda}L = 2n\pi, \quad n = 1, 2, \ldots. \tag{2}$$

However $\sin(\sqrt{\lambda}L) = 0$ for these $\sqrt{\lambda}$, i.e., $A = B = 0$ in this case.

Next let us suppose $\sin(\sqrt{\lambda}L) = 0$, or $\sqrt{\lambda}L = n\pi$ $(n = 1, 2, \ldots)$. In this case (1) becomes

$$A = A\cos(\sqrt{\lambda}), \quad B = B\cos(\sqrt{\lambda}).$$

In order to have nonzero A, B, we obtain (2). Finally we obtain [1]

$$\phi_n(x) = A\cos\frac{2n\pi x}{L} + B\sin\frac{2n\pi x}{L}, \quad \lambda_n = \left(\frac{2n\pi}{L}\right)^2, \quad n = 1, 2, \ldots,$$

and [1] $\phi_0(x) = C$, $\lambda_0 = 0$, where A, B, C are arbitrary constants.

Alternative Solution We can also focus on two cases where $1 - \cos(\sqrt{\lambda}L) \neq 0$ and $= 0$. In the former case we can proceed as

$$A = \frac{\sin(\sqrt{\lambda}L)}{1 - \cos(\sqrt{\lambda}L)} B, \quad B = \frac{-\sin(\sqrt{\lambda}L)}{1 - \cos(\sqrt{\lambda}L)} A \quad \Rightarrow \quad (2).$$

Alternative Solution We can write (1) as

$$M \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad M = \begin{pmatrix} 1 - \cos(\sqrt{\lambda}L) & -\sin(\sqrt{\lambda}L) \\ \sin(\sqrt{\lambda}L) & 1 - \cos(\sqrt{\lambda}L) \end{pmatrix}.$$

Then, "nonzero A, B" $\Leftrightarrow$ $\det M = 0$ $\Leftrightarrow$ (2).