Chapter 6
Method of characteristics

First order PDEs

First-order partial differential equations are generally written as

$$\begin{cases} a(x,t)u_t + b(x,t)u_x + c(x,t)u = 0, & -\infty < x < \infty, \quad 0 < t < \infty, \\ u(x,t) = f(x), & -\infty < x < \infty, \quad t = 0. \end{cases}$$

For simplicity we consider the following problem.

$$\begin{cases} u_t + vu_x + \alpha u = 0, & -\infty < x < \infty, \quad 0 < t < \infty, \\ u = \sin(x), & -\infty < x < \infty, \quad t = 0, \end{cases}$$

where v, α are positive constants. We introduce new coordinates $s(x,t)$, $\tau(x,t)$. They are introduced as

$$\frac{dt}{ds} = 1, \quad \frac{dx}{ds} = v, \quad t(s=0) = 0, \quad x(s=0) = \tau.$$

That is,

$$t = s, \qquad x = vs + \tau.$$

We have

$$\frac{du}{ds} = \frac{\partial u}{\partial t}\frac{dt}{ds} + \frac{\partial u}{\partial x}\frac{dx}{ds} = u_t + vu_x.$$

We now have an ODE for $u(x,t) = u(s)$:

$$\begin{cases} \dfrac{du}{ds} + \alpha u = 0, & s > 0, \\ u = \sin(\tau), & s = 0. \end{cases}$$

We obtain

$$u(s) = \sin(\tau)e^{-\alpha s}.$$

Since $s = t$, $\tau = x - vt$, we finally obtain

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Boundary Value Problems for Partial Differential Equations

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$$u(x,t) = \sin(x - vt)e^{-\alpha t}.$$

As we saw, τ is associated with the initial condition. If we trace a point $x(s,\tau)$ in the xt-plane, the point propagates along a line (curve), which is called a characteristic. Then $s \in (0,\infty)$ is a parameter along the characteristic.

Wave equation revisited

Let's attack second-order PDEs using the idea of characteristics. Consider

$$Au_{tt} + Bu_{tx} + Cu_{xx} = 0, \tag{6.1}$$

with constants A, B, C. We introduce

$$\tau_1 = x - \alpha_1 t, \qquad \tau_2 = x - \alpha_2 t.$$

Then,

$$\begin{aligned} u_t &= \frac{\partial u}{\partial \tau_1}\frac{\partial \tau_1}{\partial t} + \frac{\partial u}{\partial \tau_2}\frac{\partial \tau_2}{\partial t} = -\alpha_1 u_{\tau_1} - \alpha_2 u_{\tau_2}, \\ u_x &= u_{\tau_1} + u_{\tau_2}. \end{aligned}$$

With these τ_1 and τ_2, (6.1) is written as

$$\left(A\alpha_1^2 - B\alpha_1 + C\right) u_{\tau_1\tau_1} + \left(A\alpha_2^2 - B\alpha_2 + C\right) u_{\tau_2\tau_2} + 2\left(\alpha_1\alpha_2 A - \frac{\alpha_1+\alpha_2}{2}B + C\right) u_{\tau_1\tau_2} = 0.$$

Note that $A\alpha^2 - B\alpha + C = 0$ is solved as

$$\alpha = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}.$$

If $B^2 - AC = 0$ (parabolic) or $B^2 - AC < 0$ (elliptic), then we have one real solution or two complex solutions and we cannot simplify (6.1). If $B^2 - AC > 0$ (hyperbolic), we have two real solutions α_1 and α_2. By choosing $\alpha_{1,2} = \left(B \pm \sqrt{B^2 - 4AC}\right)/2A$, (6.1) becomes

$$\frac{4AC - B^2}{A} u_{\tau_1\tau_2} = 0.$$

That is,

$$u_{\tau_1\tau_2} = 0 \quad \Leftrightarrow \quad u = g_1(\tau_1) + g_2(\tau_2), \tag{6.2}$$

where g_1, g_2 are some functions.

We obtain
$$u(x,t) = g_1(x-\alpha_1 t) + g_2(x-\alpha_2 t).$$

We have seen this form in (5.10). In the xt-plane, $x-\alpha_1 t = \tau_1$ and $x-\alpha_2 t = \tau_2$ are called characteristics.

Note that $\frac{\partial g_1}{\partial \tau_2} = 0$, $\frac{\partial g_2}{\partial \tau_1} = 0$ from (6.2). If we view g_1 as a function of x,t, we have

$$\frac{\partial g_1}{\partial \tau_2} = \frac{\partial g_1}{\partial t}\frac{\partial t}{\partial \tau_2} + \frac{\partial g_1}{\partial x}\frac{\partial x}{\partial \tau_1} = \frac{1}{\alpha_1-\alpha_2}\left(\frac{\partial g_1}{\partial t} + \alpha_1 \frac{\partial g_1}{\partial x}\right) = 0,$$

where we used
$$t = \frac{\tau_1 - \tau_2}{\alpha_2 - \alpha_1}, \qquad x = \frac{\alpha_2 \tau_1 - \alpha_1 \tau_2}{\alpha_2 - \alpha_1},$$

Similarly we can compute $\partial g_2/\partial \tau_2$. Hence we have

$$g_{1t} + \alpha_1 g_{1x} = 0, \qquad g_{2t} + \alpha_2 g_{2x} = 0.$$

Thus a second-order hyperbolic PDE is reduced to two first-order PDEs.

Example 1.
$$u_{tt} = c^2 u_{zz}. \tag{6.3}$$

In this case, $A = 1$, $B = 0$, and $C = -c^2$. We obtain $\alpha_{1,2} = \pm c$. (6.3) reduces to $g_{1t} + c g_{1z} = 0$ and $g_{2t} - c g_{2z} = 0$. We obtain

$$u(z,t) = g_1(z-ct) + g_2(z+ct),$$

where g_1, g_2 are determined by initial conditions.