Chapter 2
PDEs in rectangular coordinates

The heat equation [1]

Let us consider the rate $\mathbf{q}(\mathbf{x},t)$ of heat flow at $\mathbf{x}=(x,y,z)$. This $\mathbf{q}$ is called the heat flux (the heat current density)[2]. Fourier's law of heat conduction states that

$$\mathbf{q}=-k\nabla u,$$

where k is the thermal conductivity of the material[3] and $u(\mathbf{x},t)$ is the temperature measured at the point $\mathbf{x}$ at the time t.

Definition 1. Consider a bounded domain R with boundary ∂R. A unit vector $\mathbf{n}(x)$ ($x\in\partial R$) is called the unit outward normal vector or the outer unit normal vector if the vector is orthogonal to the tangent vector at $x\in\partial R$ (i.e., $\mathbf{n}$ is perpendicular to the boundary ∂R) and is pointing outward.

Suppose heat is generated by internal sources at rate $s(\mathbf{x},t)$. Consider the heat Q that enters region R within the time interval $(t,t+\Delta t)$.

$$Q=\left(-\int_{\partial R}\mathbf{q}\cdot\mathbf{n}dS+\int_R sdV\right)\Delta t,$$

where $\mathbf{n}$ is the unit outward normal vector. This Q raises the temperature by $u_t\Delta t$:

$$Q=\int_R c\rho u_t dV\,\Delta t,$$

where c is the heat capacity per unit mass and ρ is the mass density of the material[4]. Therefore, we obtain

$$\int_R c\rho u_t dV=-\int_{\partial R}\mathbf{q}\cdot\mathbf{n}dS+\int_R sdV.$$

Winter 2014 Math 454 Sec 2
Boundary Value Problems for Partial Differential Equations

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[1] This section corresponds to §2.1 of the textbook.
[2] In SI units, $[\mathbf{q}]=\frac{\mathrm{J/s}}{\mathrm{m}^2}$.
[3] In SI units, $[k]=\frac{\mathrm{J/s}}{\mathrm{mK}}$.
[4] In SI units, $[c]=\frac{\mathrm{J}}{\mathrm{kgK}}$ and $[\rho]=\frac{\mathrm{kg}}{\mathrm{m}^3}$.

Note that by divergence theorem[5]

$$\int_{\partial R} \mathbf{q} \cdot \mathbf{n} dS = \int_R \nabla \cdot \mathbf{q} dV,$$

and the Laplacian $\Delta = \nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$. We thus obtain the continuity equation, which describes the conservation of energy:

$$\frac{\partial (c\rho u)}{\partial t} + \nabla \cdot \mathbf{q} = s.$$

We introduce the thermal diffusivity $K = k/c\rho$ and define $r(\mathbf{x},t) = s(\mathbf{x},t)/c\rho$.

We obtain the heat equation as

$$u_t = K\Delta u + r. \tag{2.1}$$

Equation (2.1) is also called the diffusion equation. Consider diffusion of the density u of milk dropped in coffee. Let $\mathbf{q}$ be the flux of the milk. With the continuity equation $\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{q} = 0$ and Fick's law $\mathbf{q} = -D\nabla u$, where D is the diffusion constant. We obtain[6] $u_t = D\Delta u$.

Steady state

If u is independent of t ($u_t = 0$), then $\Delta u = -r/K$ becomes Poisson's equation. Furthermore if $s = 0$, then the heat equation becomes $\Delta u = 0$, which is Laplace's equation.

Example 1. Let us find the steady-state solution of the heat equation $u_t = K\nabla^2 u$ in the slab $0 < z < L$ with the boundary conditions $u(x,y,0) = T_1$ and $u(x,y,L) = T_2$.

The steady-state solution satisfies $\nabla^2 u = 0$. We can write $u(x,y,z) = U(z)$. Hence, the general solution is obtained as $U = A + Bz$ with constants A, B. We obtain $A = T_1$ and $B = (T_2 - T_1)/L$. Finally, we obtain

$$u(\mathbf{x}) = T_1 \left(1 - \frac{z}{L}\right) + T_2 \frac{z}{L}.$$

[5] For example, Math 255.

[6] In general the diffusion coefficient D depends on $\mathbf{x}$ and $u_t = \nabla \cdot (D(\mathbf{x})\nabla u)$.

Homogeneous boundary conditions on a slab [7]

There are three ways to impose boundary conditions.

Dirichlet boundary condition: The temperature u on the boundary is given.

$$u = g(x), \quad x \in \partial\Omega.$$

Neumann boundary condition: The heat flux $\mathbf{q} = -k\nabla u$ across the boundary is given.

$$\mathbf{n}\cdot\nabla u = g(x), \quad x \in \partial\Omega.$$

Robin boundary condition: We take a linear combination of the above two boundary conditions. Newton's law of cooling $\mathbf{n}\cdot\nabla u = h(T-u)$ $(h>0)$ has this form.

$$a(x)u + b(x)\mathbf{n}\cdot\nabla u = g(x), \quad x \in \partial\Omega.$$

When $g=0$, we say the boundary condition is homogeneous (the PDE might not be homogeneous).

Let us consider homogeneous boundary conditions. In the case of a slab $0<z<L$, the condition at $x=0$ is expressed as

$$u(0,t)\cos\alpha - Lu_z(0,t)\sin\alpha = 0, \quad 0\le\alpha<\pi. \tag{2.2}$$

For $\alpha=0$, we have $u(0,t)=0$. For $\alpha=\pi/2$, we have $u_z(0,t)=-\mathbf{n}\cdot\nabla u(0,t)=0$. For $\alpha\ne 0$ we can rewrite (2.2) as $u(0,t)c\cot\alpha - cLu_z(0,t)=0$, where c is a constant. By setting $a(0)=c\cot\alpha$, $b(0)=cL$, we have $a(0)u(0,t)+b(0)\mathbf{n}\cdot\nabla u=0$. Therefore by (2.2) we can express any boundary condition at $x=0$. Thus we can write the heat equation in the slab $0<z<L$ as follows.

$$\begin{cases} u_t = Ku_{zz}, & 0<z<L, \quad t>0, \\ u\cos\alpha - Lu_z\sin\alpha = 0, & z=0, \quad t>0, \\ u\cos\beta + Lu_z\sin\beta = 0, & z=L, \quad t>0, \\ u = f(z), & 0<z<L, \quad t=0, \end{cases}$$

where $f(z)$, $0<z<L$ is a piecewise smooth function.

Let us solve the heat equation in a simple case of $\alpha=\beta=0$. The separated solution is written as $u(z,t)=\phi(z)T(t)$. Thus we obtain

$$T'(t)+\lambda KT(t)=0, \quad \phi''(z)+\lambda\phi(z)=0.$$

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[7] This section corresponds to §2.2 of the textbook.

The boundary conditions are written as

$$\phi(0) = \phi(L) = 0.$$

We obtain

$$T(t) = e^{-\lambda K t}, \quad \phi = A\sin(\sqrt{\lambda}z) + B\cos(\sqrt{\lambda}z), \qquad \lambda > 0.$$

By plugging $\phi = A\sin(\sqrt{\lambda}z) + B\cos(\sqrt{\lambda}z)$ into the boundary conditions, we find that $B = 0$ and $\sqrt{\lambda}L$ is an integer multiple of π. Therefore we obtain

$$\phi(z) = \phi_n(z) = \sin(\sqrt{\lambda_n}z), \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots,$$

where we set the arbitrary constant in $\phi_n(z)$ to be 1 (recall we will take a superposition). Thus the separated solutions are obtained as

$$u(z,t) = \phi_n(z)e^{-\lambda_n K t}, \quad n = 1, 2, \ldots.$$

If no initial condition is given, the above separated solutions are the solutions to the problem. Here, however, we have an initial condition.

Let us consider the initial condition. We express the solution as

$$u(z,t) = \sum_{n=1}^{\infty} C_n \phi_n(z) e^{-\lambda_n K t},$$

where C_n are constants. Since f is piecewise smooth, we can write $f(z) = \sum_{n=1}^{\infty} B_n \phi_n(z)$, $0 < z < L$, with some coefficients B_n (Fourier sine series). This implies that $C_n = B_n$. We obtain

$$u(z,t) = \sum_{n=1}^{\infty} B_n \phi_n(z) e^{-\lambda_n K t}, \qquad 0 < z < L, \quad t > 0. \tag{2.3}$$

Example 2. The heat equation $u_t = Ku_{zz}$ for $0 < z < L$, $t > 0$ with $u(0,t) = u(L,t) = 0$ and $u(z,0) = 1$ is solved as

$$u(z,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi z}{L} e^{-(n\pi/L)^2 K t},$$

where

$$B_n = \frac{2}{\pi}\frac{1-(-1)^n}{n}. \tag{2.4}$$

As is mentioned above, B_n are coefficients of the Fourier sine series of $f(z)$. We can also compute B_n as follows. First we note the following orthogonality relations. For $n, m = 1, 2, \ldots$, we have

$$\int_0^L \phi_n(z)\phi_m(z)dz = \int_0^L \sin\frac{n\pi z}{L}\sin\frac{m\pi z}{L}dz$$
$$= \frac{1}{2}\int_0^L \left[\cos\frac{(n-m)\pi z}{L} - \cos\frac{(n+m)\pi z}{L}\right] dz,$$
$$= \frac{L}{2}\delta_{nm}. \tag{2.5}$$

Note that the interval for the integral is $(0,L)$ instead of $(-L,L)$. The next example explains how B_n are computed using these orthogonality relations.

Example 3. Let us consider $f(z) = 1$, $0 < z < L$. We write $f(z)$ using unknown constants B_n as

$$f(z) = \sum_{n=1}^{\infty} B_n\phi_n(z).$$

By integrating both sides after multiplying $\phi_n(z)$, we obtain

$$\int_0^L f(z)\phi_n(z)dz = \int_0^L \sum_{m=1}^{\infty} B_m\phi_m(z)\phi_n(z)dz.$$

The left-hand side is calculated as

$$\text{LHS} = \int_0^L \sin\frac{n\pi z}{L}dz = \frac{-1}{n\pi}\cos\frac{n\pi z}{L}\bigg|_0^L = \frac{L}{n\pi}\left(1-(-1)^n\right).$$

Using the orthogonality relations, the right-hand side is computed as

$$\text{RHS} = \sum_{m=1}^{\infty} B_m \int_0^L \phi_m(z)\phi_n(z)dz = \sum_{m=1}^{\infty} B_m\frac{L}{2}\delta_{nm} = \frac{L}{2}B_n.$$

Therefore we obtain (2.4).

Orthogonal functions [8]

Definition 2 (Inner product). We extend dot product $\varphi\cdot\psi$ and define inner product as

$$\langle\varphi,\psi\rangle = \int_a^b \varphi(x)\psi(x)dx.$$

Sometimes the inner product is defined as follows. We can have a weight function, and the weighted inner product is given by

$$\langle\varphi,\psi\rangle_\rho = \int_a^b \varphi(x)\psi(x)\rho(x)dx,$$

[8] This section corresponds to §0.3 of the textbook.

where $\rho(x) > 0$ is a weight function. For complex functions, we can write the complex inner product as

$$\langle \varphi, \psi \rangle = \int_a^b \varphi(x)\overline{\psi}(x)dx.$$

Here $\overline{\psi}$ is the complex conjugate of ψ ($\overline{\psi}(x) = f(x) - ig(x)$ when $\psi = f + ig$).

Definition 3 (Orthogonal). Two functions φ, ψ are said to be orthogonal on $[a,b]$ if $\langle \varphi, \psi \rangle = 0$.

Example 4. The functions $\varphi(x) = \sin x$ and $\psi(x) = \cos x$ are orthogonal on $[0, \pi]$.

Example 5. The set of functions $\sin x, \sin 2x, \ldots, \sin Nx$ is orthogonal on $[0, \pi]$.

Example 6. Which of the following pairs of functions are orthogonal on the interval $0 \le x \le 1$?

$$\varphi_1 = \sin 2\pi x, \quad \varphi_2 = x, \quad \varphi_3 = \cos 2\pi x, \quad \varphi_4 = 1.$$

$\langle \varphi_1, \varphi_3 \rangle = 0$, $\langle \varphi_1, \varphi_4 \rangle = 0$, $\langle \varphi_2, \varphi_3 \rangle = 0$, $\langle \varphi_3, \varphi_4 \rangle = 0$. All others are nonzero. Therefore the pairs (φ_1, φ_3), (φ_1, φ_4), (φ_2, φ_3), and (φ_3, φ_4) are orthogonal.

Definition 4 (Norm). As follows we define norm, which is the "length" of a function.

$$\|\varphi\| = \|\varphi\|_{L^2(a,b)} = \sqrt{\langle \varphi, \varphi \rangle}.$$

We note that the norm is always nonnegative. The norm $||\varphi - \psi||$ is the distance between two functions φ and ψ.

Definition 5 (Projection). Let $(\varphi_1, \ldots, \varphi_N)$ be a set of orthogonal functions with $||\varphi_i|| \neq 0$. Let $f(x)$ be a function. Then $\hat{c}_1\varphi_1 + \cdots + \hat{c}_N\varphi_N$ with $\hat{c}_i = \langle f, \varphi_i \rangle / ||\varphi_i||^2$ is the projection of f onto $(\varphi_1, \ldots, \varphi_N)$. $\hat{c}_i$ is called the ith Fourier coefficient of f.

Note that the minimum of $||f - (c_1\varphi_1 + \cdots + c_N\varphi_N)||$ is achieved when $c_i = \hat{c}_i$.

Example 7. Find the projection of $f(x) = 1$ onto $(\varphi_1, \varphi_2) = (\sin x, \sin 2x)$ on the interval $0 \le x \le \pi$. By $\langle f, \varphi_1 \rangle = 2$, $\langle f, \varphi_2 \rangle = 0$, and $||\varphi_1||^2 = ||\varphi_2||^2 = \frac{\pi}{2}$, we obtain $\frac{4}{\pi} \sin x$.

Definition 6 (Orthonormal). The functions $(\varphi_1, \ldots, \varphi_N)$ are orthonormal if $\langle \varphi_i, \varphi_j \rangle = \delta_{ij}$. Here δ_{ij} is the Kronecker delta ($\delta_{ij} = 0$ if $i \neq j$ and $= 1$ if $i = j$).

Sturm-Liouville eigenvalue problems [9]

We saw in (2.3) that the solution was given as a Fourier series with sine functions, which are orthogonal to each other. Sometimes series with other functions appear. In this section we will develop the general theory for such orthogonal functions.

Let us begin by

$$\begin{cases} \phi''(x) + \lambda\phi(x) = 0, \quad x \in (0, L), \\ \phi(0) = \phi(L) = 0. \end{cases} \tag{2.6}$$

The nontrivial solutions[10] to (2.6) are obtained as $\phi(x) = \sin(\sqrt{\lambda}x)$ with $\lambda = (n\pi/L)^2$ ($n = 1, 2, \ldots$). We call λ and ϕ an eigenvalue and an eigenfuction of the Sturm-Liouville eigenvalue problem[11].

In general an equation for ϕ is given on the interval (a, b) and boundary conditions are given by

$$\phi(a)\cos\alpha - L\phi'(a)\sin\alpha = 0, \quad \phi(b)\cos\beta + L\phi'(b)\sin\beta = 0, \tag{2.7}$$

where $L = b - a$, and $\alpha, \beta \in [0, \pi)$ are some parameters. The most general form of Sturm-Liouville problems is written as

$$[s(x)\phi'(x)]' + [\lambda\rho(x) - q(x)]\phi(x) = 0, \quad a < x < b, \tag{2.8}$$

where $\rho(x) > 0$.

Theorem 1 (Orthogonality). *Consider the Sturm-Liouville problem (2.8) with the boundary conditions (2.7). Suppose that $\phi_1(x), \phi_2(x)$ are nontrivial solutions with different eigenvalues $\lambda_1 \neq \lambda_2$. Then the eigenfunctions are orthogonal with respect to the weight function $\rho(x)$, $a < x < b$:*

$$\int_a^b \phi_1(x)\phi_2(x)\rho(x)dx = 0.$$

If the two eigenfunctions belong to the same eigenvalue $\lambda_1 = \lambda_2$, then the eigenfunctions must be proportional:

$$\phi_2(x) = C\phi_1(x)$$

for some constant C.

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[9] This section corresponds to §1.6 of the textbook.

[10] Obviously $\phi(x) = 0$ is a solution to (2.6). This solution is called the trivial solution. Other solutions are called nontrivial solutions.

[11] Using $\phi''(x) \approx [\phi(x+\Delta x) - 2\phi(x) + \phi(x-\Delta x)]/(\Delta x)^2$, we can write (2.6) as a matrix-vector equation. Then λ becomes an eigenvalue of a matrix and ϕ becomes an eigenvector.

Theorem 2. *Let $\phi(x) \in \mathbb{C}$ and $s(x), \rho(x), q(x) \in \mathbb{R}$ in (2.8). Then λ is a real number.*

Example 8 (Bessel functions). By setting $s(x) = \rho(x) = x^{d-1}$, $q(x) = \mu x^{d-3}$, and $\lambda = 1$ with $a = 0$ and $b = \infty$, we obtain

$$\phi'' + (d-1)\frac{\phi'}{x} + \left(1 - \frac{\mu}{x^2}\right)\phi = 0,$$

where d is the dimension and μ is the angular index. In the case of $d = 2$ and $\mu = m^2$, the function $\phi(x) = J_m(x)$ is called the Bessel function[12]. In the case of $d = 3$ and $\mu = k(k+1)$ $(k = 0, 1, 2, \dots)$, the function $\phi(x) = j_k(x)$ is called the spherical Bessel function [13]..

Example 9 (Legendre polynomials). By setting $s(x) = 1 - x^2$, $\rho(x) = 1$, $q(x) = m^2/s(x)$, and $\lambda = k(k+1)$ $(k = 0, 1, 2, \dots)$ with $a = -1$ and $b = 1$, we obtain

$$(1-x^2)\phi'' - 2x\phi' + \left(k(k+1) - \frac{m^2}{1-x^2}\right)\phi = 0.$$

The function $\phi(x) = P_k^m(x)$ is called the associated Legendre polynomial[14]. When $m = 0$, the function $P_k(x)$ is called the Legendre polynomial.

Example 10 (Hermite polynomials). By setting $s(x) = \rho(x) = \exp\left(-x^2/2\right), q(x) = 0, \lambda = n$ $(n = 0, 1, 2, \dots)$ with $a = -\infty$ and $b = \infty$, we obtain

$$\phi'' - x\phi' + n\phi = 0.$$

The function $\phi(x) = H_n(x)$ is called the Hermite polynomial[15].

Proof (Theorem 1).
Let us begin with

$$\left[s(x)\phi_1'(x)\right]' + \left[\lambda_1\rho(x) - q(x)\right]\phi_1(x) = 0, \quad \left[s(x)\phi_2'(x)\right]' + \left[\lambda_2\rho(x) - q(x)\right]\phi_2(x) = 0. \tag{2.9}$$

We multiply the first equation by $\phi_2(x)$ and the second equation by $\phi_1(x)$, and integrate them.

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[12] Bessel functions appear in §3.2 of the textbook.
[13] spherical Bessel functions appear in §4.2 of the textbook.
[14] Legendre polynomials and associated Legendre polynomials appear in §4.2 of the textbook.
[15] Hermite polynomials appear in §5.2 of the textbook.

$$\begin{cases} \displaystyle\int_a^b \phi_2(x)\left[s(x)\phi_1'(x)\right]' dx + \int_a^b \phi_2(x)\left[\lambda_1\rho(x) - q(x)\right]\phi_1(x)dx = 0, \\ \displaystyle\int_a^b \phi_1(x)\left[s(x)\phi_2'(x)\right]' dx + \int_a^b \phi_1(x)\left[\lambda_2\rho(x) - q(x)\right]\phi_2(x)dx = 0. \end{cases}$$

By integration by parts, we obtain

$$\begin{cases} \left[\phi_2(x)s(x)\phi_1'(x)\right]_a^b - \displaystyle\int_a^b \phi_2'(x)s(x)\phi_1'(x)dx + \int_a^b \phi_2(x)\left[\lambda_1\rho(x) - q(x)\right]\phi_1(x)dx = 0, \\ \left[\phi_1(x)s(x)\phi_2'(x)\right]_a^b - \displaystyle\int_a^b \phi_1'(x)s(x)\phi_2'(x)dx + \int_a^b \phi_1(x)\left[\lambda_2\rho(x) - q(x)\right]\phi_2(x)dx = 0. \end{cases}$$

We subtract these equations. Noting that $\phi_1(x)$ and $\phi_2(x)$ satisfy (2.7), we have

$$\begin{aligned} &\left[\phi_2(x)s(x)\phi_1'(x)\right]_a^b - \left[\phi_1(x)s(x)\phi_2'(x)\right]_a^b \\ &= s(b)\left[\phi_2(b)\phi_1'(b) - \phi_1(b)\phi_2'(b)\right] - s(a)\left[\phi_2(a)\phi_1'(a) - \phi_1(a)\phi_2'(a)\right] \\ &= s(b)[0] - s(a)[0] = 0. \end{aligned} \tag{2.10}$$

Therefore, we obtain

$$(\lambda_1 - \lambda_2)\int_a^b \phi_1(x)\phi_2(x)\rho(x)dx = 0. \tag{2.11}$$

Since $\lambda_1 \neq \lambda_2$, this integral must be zero.

Next, let us suppose $\lambda_1 = \lambda_2 = \lambda$. We consider

$$\psi(x) = \begin{cases} \phi_2(a)\phi_1(x) - \phi_1(a)\phi_2(x), & \text{if } \alpha \neq 0, \\ \phi_2'(a)\phi_1(x) - \phi_1'(a)\phi_2(x), & \text{if } \alpha = 0. \end{cases}$$

This $\psi(x)$ obeys (2.8). We have $\psi(a) = 0$. Also $\psi'(a) = 0$ by (2.7). Thus $\psi(x)$ is a solution to (2.8) with initial conditions $\psi(a) = \psi'(a) = 0$. We can conclude that $\psi(x) = 0$, $a < x < b$. Therefore,

$$\phi_2(x) = C\phi_1(x), \qquad C = \frac{\phi_2(a)}{\phi_1(a)} \quad \text{or} \quad \frac{\phi_2'(a)}{\phi_1'(a)}.$$

□

Proof (Theorem 2). Let us set $\lambda_1 = \lambda$, $\phi_1(x) = \phi(x)$, $\lambda_2 = \overline{\lambda}$, and $\phi_2(x) = \overline{\phi}(x)$ in (2.9). Then instead of (2.11), we have

$$(\lambda - \overline{\lambda})\int_a^b |\phi(x)|^2\rho(x)dx = 0.$$

Therefore, $\lambda = \overline{\lambda}$. The imaginary part of λ is zero. □

Remark 1. The first half of the proof of Theorem 1 holds if (2.10) is verified. Thus the orthogonality in Theorems 1 can be extended to the case of periodic boundary conditions $\phi(a) = \phi(b)$, $\phi'(a) = \phi'(b)$, $s(a) = s(b)$, and the singular case of $s(a) = s(b) = 0$. We can similarly extend Theorem 2.

Just like the usual Fourier series, we can express a function $f(x)$ as

$$f(x) \simeq f_N(x), \qquad f_N(x) = \sum_{n=1}^{N} \hat{c}_n \phi_n(x), \qquad \hat{c}_n = \frac{\langle f, \phi_n \rangle_\rho}{\|\phi_n\|^2}. \tag{2.12}$$

Theorem 3 (Convergence). *Let f be a function such that $\int_a^b f(x)^2 \rho(x) dx < \infty$. (i) We have*

$$\int_a^b f_N(x)^2 \rho(x) dx \quad \to \quad \int_a^b f(x)^2 \rho(x) dx \qquad as \quad N \to \infty.$$

(ii) Furthermore we assume f is piecewise smooth. Then

$$f_N(x) \to \frac{1}{2}\left[f(x+0) + f(x-0)\right] \quad on \quad x \in (a,b) \qquad as \quad N \to \infty.$$

(iii) If $f \in C[a,b]$ (continuous on $[a,b]$), f' is piecewise continuous, and f satisfies the boundary conditions of the Sturm-Liouville problem, then f_N converges uniformly on $[a,b]$.

Remark 2. By the convergence (i), we could consider the mean square error and the convergence rate of the Fourier sine and cosine series. The convergence (ii) was seen in Figs. 1.1 and 1.2. Due to the convergence (iii), the Fourier cosine series in Fig. 1.2 converged uniformly.

Remark 3. Theorem 3(ii) can be readily extended to the interval $[a,b]$ by using $\bar{f}$ which is the periodic extension of f.

Remark 4. In the series (2.12) we labeled nontrivial solutions ϕ_n using $n = 1, 2, \ldots$. But we can rename them as $\phi_0, \phi_1, \ldots$ if it is more convenient.

Nonhomogeneous boundary conditions [16]

Let us consider the heat equation in the slab $0 < z < L$:

[16] This section corresponds to §2.3 of the textbook.

$$
\begin{cases}
u_t = K u_{zz} + r(z), & t > 0, \quad 0 < z < L, \\
u(0,t)\cos\alpha - L u_z(0,t)\sin\alpha = T_1, & t > 0, \\
u(L,t)\cos\beta + L u_z(L,t)\sin\beta = T_2, & t > 0, \\
u(z,0) = f(z), & 0 < z < L,
\end{cases}
\tag{2.13}
$$

where $f(z)$, $0 < z < L$, is a piecewise smooth function, and $K > 0$ and $\alpha, \beta \in [0, \pi)$ are constants. The equation is nonhomogeneous because there is an internal source $r(z)$. The boundary conditions are nonhomogeneous because T_1 and T_2 are nonzero. We can solve the problem as follows.

Step 1

We find the steady-state solution $U(z)$, which obeys

$$
\begin{cases}
KU''(z) + r(z) = 0, & 0 < z < L, \\
U(0)\cos\alpha - LU'(0)\sin\alpha = T_1, \\
U(L)\cos\beta + LU'(L)\sin\beta = T_2.
\end{cases}
$$

Step 2

We rewrite the problem using $v(z,t) = u(z,t) - U(z)$.

$$
\begin{cases}
v_t = K v_{zz}, & t > 0, \quad 0 < z < L, \\
v(0,t)\cos\alpha - L v_z(0,t)\sin\alpha = 0, & t > 0, \\
v(L,t)\cos\beta + L v_z(L,t)\sin\beta = 0, & t > 0, \\
v(z,0) = f(z) - U(z), & 0 < z < L.
\end{cases}
\tag{2.14}
$$

Thus $v(z,t)$ satisfies a homogeneous equation with homogeneous boundary conditions.

Step 3

We use separation of variables, $v(z,t) = \phi(z)T(t)$, and solve (2.14). Finally, we obtain

$$
u(z,t) = U(z) + v(z,t).
$$

Step 3 can be done as follows. We obtain

$$
\phi'' + \lambda\phi = 0, \quad 0 < z < L, \qquad T' + \lambda K T = 0, \quad t > 0,
$$

where $\phi(z)$ satisfies the boundary conditions

$$
\phi(0)\cos\alpha - L\phi'(0)\sin\alpha = 0, \quad \phi(L)\cos\beta + L\phi'(L)\sin\beta = 0.
$$

Note that ϕ and λ are an eigenfunction and an eigenvalue of the Sturm-Liouville eigenproblem. In particular, we have $\int_0^L \phi_n(z)\phi_m(z)dz = 0$ $(n \neq m)$. All $v_n(z,t) = \phi_n(z)e^{-\lambda_n Kt}$ satisfy $v_t = Kv_{zz}$ and the boundary conditions in (2.14). Since $v_t = Kv_{zz}$ is homogeneous (there is no source term r), any linear combination of $v_n(z,t)$ also satisfies $v_t = Kv_{zz}$ and the boundary conditions in (2.14). We write

$$v(z,t) = \sum_n^{\infty} A_n \phi_n(z) e^{-\lambda_n Kt}. \tag{2.15}$$

At $t = 0$, by multiplying $\phi_n(z)$ and integrating over z, we obtain

$$\int_0^L v(z,0)\phi_n(z)dz = \int_0^L \left[f(z) - U(z)\right]\phi_n(z)dz.$$

Using the orthogonality relations for $\phi_n(z)$, the left-hand side becomes

$$\int_0^L v(z,0)\phi_n(z)dz = \sum_{n'}^{\infty} A_{n'} \int_0^L \phi_{n'}(z)\phi_n(z)dz = A_n \|\phi_n\|^2.$$

Thus the coefficients A_n are determined by the initial condition as

$$A_n = \frac{\int_0^L \left[f(z) - U(z)\right]\phi_n(z)dz}{\int_0^L \phi_n(z)^2 dz}. \tag{2.16}$$

With A_n in (2.16), $v(z,t)$ in (2.15) satisfies (2.14) including the initial condition. Finally, the solution is

$$u(z,t) = U(z) + \sum_n^{\infty} A_n \phi_n(z) e^{-\lambda_n Kt}. \tag{2.17}$$

Example 11. Let us solve the following heat equation in a slab.

$$\begin{cases} u_t = Ku_{zz}, & t > 0, \quad 0 < z < L, \\ u(0,t) = T_1, & t > 0, \\ u(L,t) = T_2, & t > 0, \\ u(z,0) = 1, & 0 < z < L. \end{cases}$$

Step 1: We will obtain $U(z)$ which obeys

$$U''(z) = 0, \quad 0 < z < L, \qquad U(0) = T_1, \quad U(L) = T_2.$$

The coefficients of the general solution $U(z) = A + Bz$ are determined by the boundary conditions as $A = T_1$, $B = (T_2 - T_1)/L$. Step 2: We then consider $v(z,t) = u(z,t) - U(z)$ which satisfies

$$v_t = Kv_{zz}, \quad t>0, \quad 0<z<L,$$
$$v(0,t) = v(L,t) = 0, \quad t>0, \qquad v(z,0) = 1-U(z), \quad 0<z<L.$$

Step 3: By separation of variable we write $v(z,t) = \phi(z)T(t)$. The function $T(t)$ is obtained as $T(t) = e^{-\lambda Kt}$ with the separation constant λ. The function $\phi(z)$ satisfies the Sturm-Liouville problem: $\phi'' + \lambda\phi = 0$, $0<z<L$, $\phi(0) = \phi(L) = 0$. Hence we obtain

$$\phi(z) = \sin\frac{n\pi z}{L}, \quad \lambda = \left(\frac{n\pi}{L}\right)^2, \quad n = 1,2,\ldots.$$

We can write $v(z,t)$ as

$$v(z,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi z}{L} e^{-(n\pi/L)^2 Kt}.$$

The coefficients A_n are determined by the initial condition as

$$\sum_{n=1}^{\infty} A_n \sin\frac{n\pi z}{L} = 1-U(z) = 1-T_1-\frac{T_2-T_1}{L}z.$$

Using the orthogonality relations $\int_0^L \sin(n\pi z/L)\sin(m\pi z/L)dz = (L/2)\delta_{nm}$, and the integrals $\int_0^L \sin(n\pi z/L)dz = L(1-(-1)^n)/(n\pi)$ and $\int_0^L z\sin(n\pi z/L)dz = L^2(-1)^{n+1}/(n\pi)$, we find A_n. Finally we obtain

$$u(z,t) = T_1 + \frac{T_2-T_1}{L}z + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-T_1-(-1)^n(1-T_2)}{n}\sin\frac{n\pi z}{L}e^{-(n\pi/L)^2 Kt}.$$

Example 12. Let us solve the following heat equation in a slab.

$$\begin{cases} u_t = Ku_{zz}, & t>0, \quad 0<z<L, \\ u_z(0,t) = \Phi, & t>0, \\ u_z(L,t) = \Phi, & t>0, \\ u(z,0) = 1, & 0<z<L. \end{cases}$$

Step 1: We will obtain $U(z)$ which obeys [17]

$$U''(z) = 0, \quad 0<z<L, \qquad U'(0) = U'(L) = \Phi.$$

The general solution is obtained as $U(z) = A+Bz$. From the boundary conditions, $B = \Phi$. So far, A is an arbitrary constant. Step 2: We then consider $v(z,t) = u(z,t) - U(z)$ which satisfies

$$v_t = Kv_{zz}, \quad t>0, \quad 0<z<L,$$
$$v_z(0,t) = v_z(L,t) = 0, \quad t>0, \qquad v(z,0) = 1-U(z), \quad 0<z<L.$$

[17] In the case of $u_z(0,t) = \Phi_1$, $u_z(L,t) = \Phi_2$ ($\Phi_1 \neq \Phi_2$), the temperature won't come to the steady state. Indeed it is impossible to find A,B such that $U = A+Bz$ satisfies these boundary conditions.

Step 3: By separation of variable we write $v(z,t) = \phi(z)T(t)$. The function $T(t)$ is obtained as $T(t) = e^{-\lambda K t}$ with the separation constant λ. The function $\phi(z)$ satisfies the Sturm-Liouville problem: $\phi'' + \lambda\phi = 0$, $0 < z < L$, $\phi'(0) = \phi'(L) = 0$. Hence we obtain

$$\phi(z) = \cos\frac{n\pi z}{L}, \quad \lambda = \left(\frac{n\pi}{L}\right)^2, \quad n = 0,1,2,\ldots.$$

We can write $v(z,t)$ as

$$v(z,t) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi z}{L} e^{-(n\pi/L)^2 K t}.$$

The coefficients A_n are determined by the initial condition as

$$\sum_{n=0}^{\infty} A_n \cos\frac{n\pi z}{L} = 1 - U(z) = 1 - A - \Phi z.$$

According to the Sturm-Liouville theory we have $\int_0^L \cos(n\pi z/L)\cos(m\pi z/L)dz = 0$ $(n \neq m)$. Also using the integrals $\int_0^L \cos(n\pi z/L)dz = L\delta_{n0}$, $\int_0^L z\cos(n\pi z/L)dz = ((-1)^n - 1)L^2/(n\pi)^2$ $(n \neq 0)$, $\int_0^L \cos^2(n\pi z/L)dz = L/2$ $(n \neq 0)$, we obtain

$$A_0 = 1 - A - \frac{L\Phi}{2}, \qquad A_n = 2L\Phi\frac{1-(-1)^n}{(n\pi)^2}.$$

Finally we obtain

$$u(z,t) = 1 + \left(z - \frac{L}{2}\right)\Phi + \frac{2L\Phi}{\pi^2}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2}\cos\frac{n\pi z}{L}e^{-(n\pi/L)^2 K t}.$$

Uniqueness

If we plug (2.17) into (2.13), we find that (2.17) is a solution (this check is important to avoid mistakes!). Now the question is if this $u(z,t)$ is the unique solution.

Theorem 4 (Uniqueness). *The heat equation below has a unique solution.*

$$\begin{cases} u_t = K u_{zz} + r(z), & t > 0, \quad 0 < z < L, \\ u(0,t) = T_1, & t > 0, \\ u(L,t) = T_2, & t > 0, \\ u(z,0) = f(z), & 0 < z < L. \end{cases}$$

Proof. Suppose that u_1 and u_2 are solutions. We set $u = u_1 - u_2$. Then we have $u_t = Ku_{zz}$, $u(0,t) = u(L,t) = 0$, and $u(z,0) = 0$. By multiplying u on both sides, we have

$$u_t = Ku_{zz} \quad \Rightarrow \quad uu_t = Kuu_{zz} \quad \Rightarrow \quad \frac{1}{2}\partial_t u^2 = K\left(\frac{1}{2}\partial_z^2 u^2 - u_z^2\right).$$

By integrating both sides, we obtain

$$\frac{1}{2}\partial_t \int_0^L u^2 dz = K\left(\frac{1}{2}\ \partial_z u^2\big|_0^L - \int_0^L u_z^2 dz\right) = K\ uu_z|_0^L - K\int_0^L u_z^2 dz.$$

We introduce

$$w(t) = \frac{1}{2}\int_0^L u(z,t)^2 dz.$$

We have

$$w'(t) = -K\int_0^L u_z(z,t)^2 dz.$$

Thus we have

$$w(t) \geq 0, \qquad w'(t) \leq 0.$$

The initial condition $u(z,0) = 0$ implies

$$w(0) = 0.$$

Therefore $w(t) = 0$ $(t \geq 0)$. This then implies $u_1 = u_2$ $(t \geq 0,\ 0 \leq z \leq L)$. □

Note that[18] the boundary conditions were used only to show $uu_z|_0^L = 0$ in the above proof. Hence the theorem can be extended to other boundary conditions such as $u_z(0) = u_z(L) = 0$.

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[18] We can also prove the uniqueness for the general boundary conditions $u(0,t)\cos\alpha - Lu_z(0,t)\sin\alpha = T_1$ and $u(L,t)\cos\beta + Lu_z(L,t)\sin\beta = T_2$ with $\alpha,\beta \in (0,\pi)$. In this case $u = u_1 - u_2$ satisfies $u(0,t)\frac{\cot\alpha}{L} - u_z(0,t) = 0$ and $u(L,t)\frac{\cot\beta}{L} + u_z(L,t) = 0$. If $\alpha,\beta \leq \frac{\pi}{2}$, then $\cot\alpha, \cot\beta \geq 0$ and the above proof still holds because $uu_z|_0^L = -\frac{\cot\beta}{L}u(L)^2 - \frac{\cot\alpha}{L}u(0)^2 \leq 0$. Otherwise we can calculate the second derivative $w''(t) = \int_0^L u_t^2 dz + \int_0^L uu_{tt} dz$. The second term is calculated as $\int_0^L uu_{tt} dz = K\int_0^L u\partial_t u_{zz} dz = K\ u\partial_t u_z|_0^L - K\int_0^L u_z \partial_t u_z dz = K\ u\partial_t u_z|_0^L - K\ u_z\partial_t u|_0^L + K\int_0^L u_{zz}\partial_t u dz = K\int_0^L u_{zz}u_t dz = \int_0^L u_t^2 dz$. Hence $w''(t) = 2\int_0^L u_t^2 dz \geq 0$ and $w(t)$ is convex. Since $w(t) \geq 0$, $w''(t) \geq 0$, we see that $w(t)$ monotonically grows. We can show that $ww'' - (w')^2 = w(t)\lim_{h\to 0}\frac{w(t+h)-2w(t)+w(t-h)}{h^2} - \left[\lim_{h\to 0}\frac{w(t+h)-w(t)}{h}\right]^2 \leq \frac{1}{h^2}\lim_{h\to 0}\left\{\left[w(t+h)^2 - 2w(t)^2 + w(t)^2\right] - \left[w(t+h)^2 - 2w(t)^2 + w(t)^2\right]\right\} = 0$, where we used $w(t-h) \leq w(t) \leq w(t+h)$. Note that we have the equality only when $w(t) = 0$. On the other hand, according to the definition of $w(t)$ we obtain $ww'' - (w')^2 = \frac{1}{2}\int_0^L u^2 dz \cdot 2\int_0^L u_t^2 dz - \left(\int_0^L uu_t dz\right)^2 \geq 0$, where we used the Cauchy-Schwarz inequality. Therefore $w(t)w''(t) - (w'(t))^2 = 0$, which implies $w(t) = 0$. That is, we have proved $u = u_1 - u_2 = 0$.

Asymptotic behavior

We investigate the asymptotic behavior when $t \to \infty$.

We define the relaxation time τ as

$$\frac{1}{\tau} = -\lim_{t\to\infty} \frac{1}{t} \ln |v(z,t)|.$$

Since the above definition implies

$$v(z,t) = O\left(e^{-t/\tau}\right),$$

we see that $u(z,t)$ $(= U(z) + v(z,t))$ comes to the steady state about at $t = \tau$.

Let us consider the case of $u(0,t) = 0$, $u(L,t) = 0$. We obtain

$$v_n(z,t) = A_n \sin \frac{n\pi z}{L} e^{-(n\pi/L)^2 Kt}, \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2.$$

Let M be the maximum of A_n ($|A_n| \le M$). We have

$$\begin{aligned}
|u(z,t) - U(z)| &= \left| \sum_{n=1}^{\infty} A_n \sin \frac{n\pi z}{L} e^{-(n\pi/L)^2 Kt} \right| \le \sum_{n=1}^{\infty} |A_n| \left|\sin \frac{n\pi z}{L}\right| \left|e^{-(n\pi/L)^2 Kt}\right| \\
&\le M \sum_{n=1}^{\infty} \left(e^{-at}\right)^{(n^2)} \le M \sum_{n=1}^{\infty} \left(e^{-at}\right)^n \\
&= \frac{Me^{-at}}{1 - e^{-at}} = Me^{-at}\left(1 + e^{-at} + e^{-2at} + \dots\right) \\
&\sim Me^{-at} \quad \text{as } t \to \infty,
\end{aligned}$$

where $a = \frac{\pi^2 K}{L^2}$. Note that $S = \sum_{n=1}^{\infty} k^n = k/(1-k)$ because $kS = \sum_{n=1}^{\infty} k^{n+1} = \sum_{n=2}^{\infty} k^n = S - k$. Therefore,

$$\frac{1}{\tau} = -\lim_{t\to\infty} \frac{1}{t} \ln \left|Me^{-at}\right| = a.$$

The relaxation time τ is obtained as

$$\tau = \frac{1}{\lambda_1 K} = \frac{L^2}{\pi^2 K}.$$

Time-dependent sources and boundaries

Here we suppose that r, T_1, T_2 depend on time t. In the slab $0 < z < L$ we have

$$\begin{cases} u_t = K u_{zz} + r(z,t), & t > 0, \quad 0 < z < L, \\ u(0,t)\cos\alpha - L u_z(0,t)\sin\alpha = T_1(t), & t > 0, \\ u(L,t)\cos\beta + L u_z(L,t)\sin\beta = T_2(t), & t > 0, \\ u(z,0) = f(z), & 0 < z < L, \end{cases} \tag{2.18}$$

where $K > 0$ and $\alpha, \beta \in [0, \pi)$ are constants. We assume that $r(z,t)$, $f(z)$, $T_1(t)$, $T_2(t)$ are piecewise smooth functions.

Step 1

We fisrt solve the problem below.

$$\begin{cases} U_{zz}(z,t) = 0, & 0 < z < L, \\ U(0,t)\cos\alpha - L U_z(0,t)\sin\alpha = T_1(t), & t > 0, \\ U(L,t)\cos\beta + L U_z(L,t)\sin\beta = T_2(t), & t > 0. \end{cases}$$

Step 2

We define $v(z,t) = u(z,t) - U(z,t)$. We also introduce

$$R(z,t) = r(z,t) - U_t(z,t), \qquad F(z) = f(z) - U(z,0).$$

$$\begin{cases} v_t = K v_{zz} + R(z,t), & t > 0, \quad 0 < z < L, \\ v(0,t)\cos\alpha - L v_z(0,t)\sin\alpha = 0, & t > 0, \\ v(L,t)\cos\beta + L v_z(L,t)\sin\beta = 0, & t > 0, \\ v(z,0) = F(z), & 0 < z < L. \end{cases} \tag{2.19}$$

Step 3

We solve (2.19) for $v(z,t)$, and finally we obtain $u(z,t) = U(z,t) + v(z,t)$.

The function $U(z,t)$ in Step 1 is obtained as

$$U(z,t) = A(t) + B(t)z.$$

The coefficients A, B are independent of z, and thus U satisfies $U_{zz}(z,t) = 0$. Since U also depends on t, the coefficients A, B are functions of t in general. They are determined so that the two boundary conditions are satisfied. In matrix form, they are written as

$$\begin{pmatrix} \cos\alpha & -L\sin\alpha \\ \cos\beta & L\cos\beta + L\sin\beta \end{pmatrix} \begin{pmatrix} A(t) \\ B(t) \end{pmatrix} = \begin{pmatrix} T_1(t) \\ T_2(t) \end{pmatrix}.$$

If $\det = L(\cos\alpha\sin\beta + \sin\alpha\cos\beta + \cos\alpha\cos\beta) \neq 0$, we can find A, B uniquely. In the Sturm-Liouville eigenvalue problem below, the right-hand side is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Hence for α, β with the condition $\det \neq 0$, eigenvalues of the Sturm-Liouville problem are all nonzero.

Let us consider Step 3. We consider the following Sturm-Liouville problem.

$$\phi'' + \lambda\phi = 0, \quad 0 < z < L,$$
$$\phi(0)\cos\alpha - L\phi'(0)\sin\alpha = 0, \quad \phi(L)\cos\beta + L\phi'(L)\sin\beta = 0.$$

Let λ_n and ϕ_n be eigenvalues and eigenfunctions of this Sturm-Liouville eigenproblem. We express v, R, F with ϕ_n (recall the convergence theorem):

$$v(z,t) = \sum_{n=1}^{\infty} v_n(t)\phi_n(z), \quad R(z,t) = \sum_{n=1}^{\infty} R_n(t)\phi_n(z), \quad F(z) = \sum_{n=1}^{\infty} F_n\phi_n(z).$$

Note that $v(z,t)$ written in this way automatically satisfies the boundary conditions in (2.19). Here by using $\int_0^L \phi_n(z)\phi_m(z)dz = 0$ ($n \neq m$), we obtain

$$R_n = \frac{\int_0^L [r(z,t) - U_t(z,t)]\,\phi_n(z)dz}{\int_0^L \phi_n(z)^2 dz}, \quad F_n = \frac{\int_0^L [f(z) - U(z)]\,\phi_n(z)dz}{\int_0^L \phi_n(z)^2 dz}.$$

By substituting these expansions in (2.19), we obtain

$$\begin{cases} v_n(t)' = -\lambda_n K v_n(t) + R_n(t), \quad t > 0, \\ \hfill v_n(0) = F_n. \end{cases}$$

We can solve this equation as (recall we can always solve first-order ODEs)

$$v_n(t) = F_n e^{-\lambda_n K t} + \int_0^t R_n(s) e^{-\lambda_n K(t-s)} ds.$$

Finally we obtain

$$\begin{aligned} u(z,t) &= U(z,t) + v(z,t) \\ &= U(z,t) + \sum_{n}^{\infty} \left[F_n e^{-\lambda_n K t} + \int_0^t R_n(s) e^{-\lambda_n K(t-s)} ds \right] \phi_n(z). \end{aligned}$$

Example 13. Let us solve the following heat equation in a slab.

$$\begin{cases} u_t = Ku_{zz}, & t > 0, \quad 0 < z < L, \\ u(0,t) = 0, & t > 0, \\ u(L,t) = t, & t > 0, \\ u(z,0) = 0, & 0 < z < L. \end{cases}$$

Step 1: We will obtain $U(z,t)$ which obeys

$$U_{zz}(z,t) = 0, \quad 0 < z < L, \qquad U(0,t) = 0, \quad U(L,t) = t.$$

The coefficients of the general solution $U(z,t) = A + Bz$ are determined by the boundary conditions as $A = 0$, $B = t/L$. Step 2: We then consider $v(z,t) = u(z,t) - U(z,t)$ and $R(z,t) = -U_t(z,t) = -z/L$ which satisfies

$$\begin{aligned} &v_t = Kv_{zz} + R(z,t), \quad t > 0, \quad 0 < z < L, \\ &v(0,t) = v(L,t) = 0, \quad t > 0, \qquad v(z,0) = 0, \quad 0 < z < L. \end{aligned}$$

Step 3: We solve the following Sturm-Liouville problem.

$$\phi'' + \lambda\phi = 0, \quad 0 < z < L, \qquad \phi(0) = \phi(L) = 0.$$

We are familiar with this problem and obtain $\phi_n(z) = \sin(n\pi z/L)$, $\lambda_n = (n\pi/L)^2$ $(n = 1, 2, \dots)$. We express v, R as

$$v(z,t) = \sum_{n=1}^{\infty} v_n(t)\phi_n(z), \quad R(z,t) = \sum_{n=1}^{\infty} R_n\phi_n(z).$$

Here we obtain

$$R_n = \frac{\int_0^L (-z/L)\phi_n(z)dz}{\int_0^L \phi_n(z)^2 dz} = \frac{-2}{L^2}\int_0^L z\sin\frac{n\pi z}{L}dz = \frac{2(-1)^n}{n\pi}.$$

The coefficients $v_n(t)$ satisfy

$$v_n(t)' = -\lambda_n K v_n(t) + R_n, \quad t > 0, \qquad v_n(0) = 0.$$

This equation is solved as

$$v_n(t) = \int_0^t R_n e^{-\lambda_n K(t-s)} ds.$$

Finally we obtain

$$u(z,t) = \frac{tz}{L} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{1 - e^{-\lambda_n K t}}{\lambda_n K} \sin\frac{n\pi z}{L}.$$

Remark 5. In Example 12, we gave up solving the problem, $u_t = Ku_{zz}$ $(t > 0, 0 < z < L)$ with boundary conditions $u_z(0,t) = \Phi_1$, $u_z(L,t) = \Phi_2$ $(t > 0)$ and initial condition

$u(z,0) = 1$ $(0 < z < L)$. We can also solve this problem by slightly modifying the method which we developed for time-dependent boundaries.[19]

The wave equation [20]

Let us consider electromagnetic fields in a linear, isotropic, homogeneous medium. The fields **E** and **H** are governed by Maxwell's equations:

$$\nabla \cdot \mathbf{E} = 0, \qquad \nabla \cdot \mathbf{H} = 0,$$

$$\nabla \times \mathbf{E} = -\frac{\mu}{c}\frac{\partial \mathbf{H}}{\partial t}, \qquad \nabla \times \mathbf{H} = \frac{\varepsilon}{c}\frac{\partial \mathbf{E}}{\partial t}.$$

Here ε is the dielectric permittivity, μ is the magnetic permeability, and c is the speed of light in vacuum.

By using $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$, we obtain

$$\partial_t^2 \mathbf{E} = \frac{c^2}{\varepsilon\mu}\Delta \mathbf{E}, \qquad \partial_t^2 \mathbf{H} = \frac{c^2}{\varepsilon\mu}\Delta \mathbf{H}. \tag{2.20}$$

If we focus on one component, for example, we have

$$\frac{\partial^2 E^{(1)}}{\partial t^2} = \frac{c^2}{\varepsilon\mu}\frac{\partial^2 E^{(1)}}{\partial x^2} \tag{2.21}$$

Equations (2.20) and (2.21) are called the wave equation.

[19] With a constant C, we introduce $U(z)$ as $U_{zz}(z) = C$ $(t > 0, 0 < z < L)$, $U'(0) = \Phi_1$, $U'(L) = \Phi_2$. Then $U(z) = A + Bz + (C/2)z^2$ with $B = \Phi_1$, $C = (\Phi_2 - \Phi_1)/L$. Here A is an arbitrary constant. Then we can introduce $v(z,t)$ as $v_t = Kv_{zz} + R$ $(t > 0,\ 0 < z < L)$, $v_z(0,t) = v_z(L,t) = 0$ $(t > 0)$, $v(z,0) = F(z)$ $(0 < z < L)$, where $R = KC$, $F(z) = 1 - U(z)$. We consider the Sturm-Liouville problem, $\phi'' + \lambda\phi = 0$, $\phi'(0) = \phi'(L) = 0$. We obtain $\phi = \phi_n = \cos(n\pi z/L)$, $\lambda = \lambda_n = (n\pi/L)^2$ $(n = 0,1,\ldots)$. Let us expand v,R,F using ϕ_n: $v(z,t) = \sum_{n=0}^{\infty} v_n(t)\phi_n(z)$, $R = \sum_{n=0}^{\infty}\phi_n(z)$, and $F(z) = \sum_{n=0}^{\infty} F_n\phi_n(z)$. Note that $R_n = CK\delta_{n0}$, $F_0 = 1 - A - BL/2 - CL^2/6$, and $F_{n\geq 1} = -(2/L)\int_0^L (Bz + Cz^2/2)\phi_n(z)dz = -2L[B((-1)^n - 1) + CL(-1)^n]/(n\pi)^2$, where we used $\int_0^L z\cos(n\pi z/L)dz = L^2((-1)^n - 1)/(n\pi)^2$ and $\int_0^L z^2\cos(n\pi z/L)dz = 2L^3(-1)^n/(n\pi)^2$. Thus we have $v_0(z,t) = F_0(z) + CKt$, $v_{n\geq 1}(z,t) = F_n(z)e^{-\lambda_n Kt}$. Finally we obtain $u(z,t) = U(z) + v(z,t) = 1 + (z - L/2)\Phi_1 + (z^2 - L^2/3)(\Phi_2 - \Phi_1)/(2L) + t(\Phi_2 - \Phi_1)K/L + (2L/\pi^2)\sum_{n=1}^{\infty}(1/n^2)[\Phi_1 - (-1)^n\Phi_2]\cos(n\pi z/L)e^{-(n\pi/L)^2 Kt}$. We see that the steady state is not achieved and the temperature asymptotically behaves as $u(z,t) \sim t(\Phi_2 - \Phi_1)K/L$ as $t \to \infty$.

[20] This section corresponds to §2.4 of the textbook.

The vibrating string

Let us consider a segment of a string between $s_1 = s$ and $s_2 = s + \Delta s$, where s is the length from the left edge. The segment lies in the x-y plane between x and $x + \Delta x$. Let $y(s,t)$ be the vertical position of the string at s. Let $\rho(s)$ be the mass density function, i.e., $\int_a^b \rho(s)ds$ is the mass of the segment of the string in $s \in [a,b]$. Newton's equation of motion for the segment is written as

$$\rho(s)\Delta s \frac{\partial^2 y}{\partial t^2} = \rho(s)\Delta s f(s,t) + T_2 \sin\theta_2 - T_1 \sin\theta_1, \tag{2.22}$$

where T_1 and T_2 be the tension at s_1 and s_2, respectively, and $f(s,t) = -g$ (g is the gravitational acceleration). Since the string doesn't move in the horizontal direction along the x-axis, we have

$$T_1 \cos\theta_1 = T_2 \cos\theta_2. \tag{2.23}$$

We consider small vibrations of a string. That is, we assume that θ_1, θ_2 are small. By Taylor series we approximate $\cos\theta$ and $\sin\theta$ as

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots \simeq 1, \quad \sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots \simeq \theta.$$

By (2.23) we have $T_1 \simeq T_2$. We set

$$T_1 = T_2 = T_0.$$

The slope at each edge of the segment is

$$\left.\frac{\partial y}{\partial x}\right|_x = \tan\theta_1 \simeq \theta_1 \simeq \sin\theta_1, \qquad \left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} = \tan\theta_2 \simeq \theta_2 \simeq \sin\theta_2.$$

Thus for $|\theta| \ll 1$, the equation of motion (2.22) becomes

$$\rho \Delta s y_{tt} = \rho \Delta s f + T_0 \left[y_x(x+\Delta x) - y_x(x)\right].$$

Noting that $\min(\Delta s\cos\theta_1, \Delta s\cos\theta_2) \le \Delta x \le \max(\Delta s\cos\theta_1, \Delta s\cos\theta_2)$, and $\Delta s \simeq \Delta x$, we have

$$y_{tt} = f + \frac{T_0}{\rho}\frac{y_x(x+\Delta x) - y_x(x)}{\Delta x}.$$

Thus we obtain the wave equation:

$$\frac{\partial^2 y}{\partial t^2} = f(x,t) + \frac{T_0}{\rho(x)}\frac{\partial^2 y}{\partial x^2}.$$

The distance s along the segment is a function of x. Hence we used x in f, ρ (the new function $f(x,t), \rho(x)$ are equal to $f(s(x),t), \rho(s(x))$ in the original functions).

Example 14. Let c be a positive constant and $f(x)$, $0 < x < L$, be a piecewise smooth function. We consider the wave equation below.

$$\begin{cases} y_{tt} = c^2 y_{xx}, \quad t > 0, \quad 0 < x < L, \\ y(0,t) = y(L,t) = 0, \\ y(x,0) = f(x), \\ y_t(x,0) = 0. \end{cases}$$

By separation of variables $y(x,t) = \phi(x)T(t)$, we obtain

$$\phi'' + \lambda\phi = 0, \qquad \phi(0) = \phi(L) = 0,$$

and

$$T'' + \lambda c^2 T = 0, \qquad T'(0) = 0.$$

We can solve these equations as $\phi(x) = \phi_n(x)$, $T(t) = T_n(t)$, $n = 1, 2, \ldots$, where

$$\phi_n(x) = \sin(\sqrt{\lambda_n} x), \quad T_n(t) = \cos(\sqrt{\lambda_n} ct), \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2.$$

Thus the solution is written as

$$y(x,t) = \sum_{n=1}^{\infty} B_n \cos\frac{n\pi ct}{L} \sin\frac{n\pi x}{L}.$$

The coefficients B_n are determined by $y(x,0) = f(x)$:

$$B_n = \frac{2}{L}\int_0^L f(x)\phi_n(x)dx.$$

Alternatively we can also extend $f(x)$ as an odd function $f_O(x)$ and consider $\sum_{n=1}^{\infty} B_n \sin(n\pi x/L) = f_O(x)$ on $x \in (-L, L)$. The we have

$$B_n = \frac{1}{L}\int_{-L}^{L} f_O(x)\sin\frac{n\pi x}{L}dx = \frac{2}{L}\int_0^L f_O(x)\sin\frac{n\pi x}{L}dx = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}ds.$$

Applications of multiple Fourier series [21]

Double Fourier series appear if there are more than two variables.

[21] This section corresponds to §2.5 of the textbook.

Theorem 5 (Orthogonality relations). *For $m,n = 1,2,\cdots$, we have*

$$\int_0^{L_2}\int_0^{L_1} \sin\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2}\sin\frac{m'\pi x}{L_1}\sin\frac{n'\pi y}{L_2}dxdy = \frac{L_1L_2}{4}\delta_{mm'}\delta_{nn'}.$$

Proof. Recall (see (2.5))

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx = \frac{L}{2}\delta_{nm}.$$

□

Example 15. Consider small transverse vibrations $u(x,y,t)$ of a membrane.

$$\begin{cases} u_{tt} = c^2(u_{xx}+u_{yy}), & 0<x<L_1, \quad 0<y<L_2, \quad t>0, \\ u=0, & x=0, \quad x=L_1, \quad y=0, \quad y=L_2, \quad t>0, \\ u(x,y,0)=0, \quad u_t(x,y,0)=1, & 0<x<L_1, \quad 0<y<L_2. \end{cases}$$

We assume the following form

$$u(x,y,t) = \phi_1(x)\phi_2(y)T(t).$$

We obtain $T''/T = c^2\left[(\phi_1''/\phi_1)+(\phi_2''/\phi_2)\right]$. By introducing separation constants λ, μ_1, and μ_2, we obtain

$$T''+\lambda c^2 T = 0, \quad \phi_1''+\mu_1\phi_1 = 0, \quad \phi_2''+\mu_2\phi_2 = 0,$$

where $T''/T = -\lambda c^2$, $\phi_1''/\phi_1 = -\mu_1$, and $\phi_2''/\phi_2 = -\mu_2$. Note that

$$\lambda = \mu_1+\mu_2.$$

There are many cases: $\lambda>0$, $\lambda=0$, $\lambda<0$, $\mu_1>0$, $\mu_1=0$, $\mu_1<0$, $\mu_2>0$, $\mu_2=0$, and $\mu_2<0$. Let us consider the boundary conditions. For $\phi_1(x)$, we have

$$\phi_1''+\mu_1\phi_1 = 0, \quad \phi_1(0)=\phi_1(L_1)=0.$$

Nontrivial solutions are possible only when $\mu_1>0$. We obtain

$$\phi_1(x) = \sin\frac{m\pi x}{L_1}, \quad \mu_1 = \left(\frac{m\pi}{L_1}\right)^2, \quad m = 1,2,\ldots.$$

Note that we omitted a constant factor. Similarly for $\phi_2(y)$, we have

$$\phi_2(y) = \sin\frac{n\pi x}{L_2}, \quad \mu_2 = \left(\frac{n\pi}{L_2}\right)^2, \quad n = 1,2,\ldots.$$

Because $\lambda > 0$ ($\lambda = \mu_1 + \mu_2$), $T(t)$ is obtained as

$$T(t) = A\cos\left(\sqrt{\lambda}ct\right) + B\sin\left(\sqrt{\lambda}ct\right).$$

($u(x,y,0) = 0$ implies $T(0) = 0$ and we can easily obtain $A = 0$, but let's keep both terms here, so that we can see how A and B are determined in general.) The general solution is thus given by

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2}\left(A_{mn}\cos(\omega_{mn}t) + B_{mn}\sin(\omega_{mn}t)\right),$$

where

$$\omega_{mn} = c\sqrt{\left(\frac{m\pi}{L_1}\right)^2 + \left(\frac{n\pi}{L_2}\right)^2}.$$

Now we consider the initial conditions; $u(x,y,0) = 0$ implies $A_{mn} = 0$, and $u_t(x,y,0) = 1$ implies

$$1 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\omega_{mn}\sin\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2}.$$

Using orthogonality relations,

$$\begin{aligned}
&\int_0^{L_2}\int_0^{L_1} \sin\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2}dxdy \\
&= \int_0^{L_2}\int_0^{L_1} \sin\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2}\sum_{m'=1}^{\infty}\sum_{n'=1}^{\infty} B_{m'n'}\omega_{m'n'}\sin\frac{m'\pi x}{L_1}\sin\frac{n'\pi y}{L_2}dxdy \\
&= \sum_{m'=1}^{\infty}\sum_{n'=1}^{\infty} B_{m'n'}\omega_{m'n'}\int_0^{L_2}\int_0^{L_1}\sin\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2}\sin\frac{m'\pi x}{L_1}\sin\frac{n'\pi y}{L_2}dxdy \\
&= \sum_{m'=1}^{\infty}\sum_{n'=1}^{\infty} B_{m'n'}\omega_{m'n'}\frac{L_1L_2}{4}\delta_{mm'}\delta_{nn'} = B_{mn}\omega_{mn}\frac{L_1L_2}{4}.
\end{aligned}$$

We obtain

$$\begin{aligned}
B_{mn}\omega_{mn} &= \frac{4}{L_1L_2}\int_0^{L_1}\sin\frac{m\pi x}{L_1}dx\int_0^{L_2}\sin\frac{n\pi y}{L_2}dy \\
&= \frac{4}{L_1L_2}\frac{L_1}{m\pi}\left(1-\cos(m\pi)\right)\frac{L_2}{n\pi}\left(1-\cos(n\pi)\right) \\
&= \frac{4}{\pi^2}\frac{[1-(-1)^m][1-(-1)^n]}{mn}.
\end{aligned}$$

Finally, the solution is

$$u(x,y,t) = \frac{4}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{[1-(-1)^m][1-(-1)^n]}{mn\omega_{mn}}\sin\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2}\sin(\omega_{mn}t).$$