Chapter 1
Introduction and preliminaries

Partial differential equations [1]

What is a partial differential equation?

ODEs (Ordinary Differential Equations) have one variable (x). PDEs (Partial Differential Equations) have multiple variables ($x, y, \ldots$).

For $f(x)$ with one variable x, we know $f'(x) = \frac{df}{dx}$. For $u(x,y)$, we introduce partial derivatives as

$$\frac{\partial u}{\partial x} = \left.\frac{du}{dx}\right|_{y \text{ is fixed}} = \partial_x u = u_x.$$

Similarly,

$$\frac{\partial^2 u}{\partial x^2} = \partial_x^2 u = u_{xx}.$$

Example 1. For $u(x,y) = xy^2$, we have

$$u_x = y^2, \quad u_{xx} = 0, \quad u_y = 2xy, \quad u_{yy} = 2x.$$

Example 2. The equations below are PDEs.

$$\begin{aligned} u_{xx} - u_y &= 0 \quad \text{(the heat equation)} \\ u_{xx} - u_{yy} &= 0 \quad \text{(the wave equation)} \\ u_{xx} + u_{yy} &= 0 \quad \text{(Laplace's equation)} \end{aligned}$$

The order of a PDE is the order of the highest-order derivative in the equation. The above examples are second order. The equation $u_x + u_y = 0$ is first order.

Let us write a PDE as

$$\mathscr{L} u = g,$$

where g is independent of u. In the case of the wave equation, $\mathscr{L} = \partial_x^2 - \partial_y^2$. The equation with $\mathscr{L} = \partial_x^2 + \partial_y^2$ and some function $g(x,y)$ is called Poisson's equation.

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Boundary Value Problems for Partial Differential Equations

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[1] This section corresponds to §0.1 of the textbook.

If we have

$$\mathscr{L}(u+v)=\mathscr{L}u+\mathscr{L}v, \quad \mathscr{L}(cu)=c\mathscr{L}u,$$

for any functions u,v and constant c, then $\mathscr{L}$ is a linear differential operator and the equation is said to be linear. If $g\equiv 0$, then the equation is called homogeneous. The heat equation, wave equation, and Laplace's equation are all linear homogeneous equations. Poisson's equation is a linear nonhomogeneous PDE.

Classification of second-order PDEs

In this course we will mainly consider second-order equations. In general, second-order PDEs are written as

$$a(x,y)u_{xx}+b(x,y)u_{xy}+c(x,y)u_{yy}+d(x,y)u_x+e(x,y)u_y+f(x,y)u=g(x,y),$$

with coefficients a,b,c,d,e,f and source term g. We assume $a^2+b^2+c^2\neq 0$ (at least one of a,b,c is nonzero). These equations are classified as follows by the coefficients a,b,c.

$$\begin{cases} 4ac-b^2>0 & \text{elliptic}\\ 4ac-b^2=0 & \text{parabolic}\\ 4ac-b^2<0 & \text{hyperbolic}\end{cases}$$

Let α,β,γ be constants. Note that [2] ellipses $(x/\alpha)^2+(y/\beta)^2=1$ satisfy $4(1/\alpha^2)(1/\beta^2)-0^2>0$, parabolas $y^2=4\gamma x$ satisfy $4\cdot 0\cdot 1-0^2=0$, and hyperbolas $(x/\alpha)^2-(y/\beta)^2=1$ satisfy $4(1/\alpha^2)(-1/\beta^2)-0^2<0$.

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[2] For constants a,b,c, let us consider the symmetric matrix $A=\begin{pmatrix} a & b/2\\ b/2 & c\end{pmatrix}$. Then we have

$$au_{xx}+bu_{xy}+cu_{yy}=\sum_{i=1}^{2}\sum_{j=1}^{2}A_{ij}u_{ij}=\sum_{i=1}^{2}\sum_{j=1}^{2}A_{ij}\frac{\partial^2 u}{\partial_i\partial_j}=(\partial_x\ \partial_y)A\begin{pmatrix}\partial_x\\ \partial_y\end{pmatrix}.$$

Let $\lambda=\lambda_1,\lambda_2$ be the eigenvalues of A.

$$\det\begin{pmatrix} a-\lambda & b/2\\ b/2 & c-\lambda\end{pmatrix}=0 \quad\Leftrightarrow\quad \lambda=\frac{1}{2}\left[a+c\pm\sqrt{(a+c)^2+D}\right],$$

where $D=b^2-4ac$. Therefore,

$$\begin{array}{ccccccc} \text{elliptic} & \Leftrightarrow & D<0 & \Leftrightarrow & \lambda_1>0,\lambda_2>0 & \text{or} & \lambda_1<0,\lambda_2<0,\\ \text{parabolic} & \Leftrightarrow & D=0 & \Leftrightarrow & \lambda_1=0,\lambda_2\neq 0 & \text{or} & \lambda_1\neq 0,\lambda_2=0,\\ \text{hyperbolic} & \Leftrightarrow & D>0 & \Leftrightarrow & \lambda_1<0,\lambda_2>0 & \text{or} & \lambda_1>0,\lambda_2<0.\end{array}$$

The heat equation is parabolic and the wave equation is hyperbolic. Laplace's equation and Poisson's equation are elliptic.

Complex Numbers

Let us recall Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{1.1}$$

where $i = \sqrt{-1}$, $i^2 = -1$. We can write $\cos\theta, \sin\theta$ using exponential functions:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Note that $\cosh x$ and $\sinh x$ are defined similarly:

$$\cosh x = \frac{e^{x} + e^{-x}}{2}, \quad \sinh x = \frac{e^{x} - e^{-x}}{2}.$$

Note also the following relations.

$$\cos^2\theta + \sin^2\theta = 1, \quad \cosh^2 x - \sinh^2 x = 1,$$

$$\cos(i\theta) = \cosh\theta, \quad \sin(i\theta) = i\sinh\theta, \quad \cosh(ix) = \cos x, \quad \sinh(ix) = i\sin x.$$

Review of ODEs (1) [3]

Let us consider ODEs of $y(x)$.

Separable

Sometimes we can separate x and y in the equation. In such a case, the equation is said to be separable.

Example 3. Let us solve $y' = -6xy$. We can rewrite this as

$$\frac{dy}{y} = -6x dx.$$

[3] See A.1 of the textbook.

We integrate both sides: $\displaystyle\int \frac{dy}{y} = \int (-6x)dx$. Thus we obtain $\ln|y| = -3x^2 + C'$, and hence $y = Ce^{-3x^2}$, where $C\ (= \pm e^{C'})$ is an arbitrary constant.

Linear first-order equations

Consider

$$y'(x) + p(x)y(x) = q(x), \quad y(x_0) = y_0.$$

This type can always be solved as follows. By multiplying the integrating factor $\exp\left(\int_{x_0}^{x} p(x')dx'\right)$, we have $\frac{d}{dx}\left[y(x)e^{\int_{x_0}^{x} p(x')dx'}\right] = q(x)e^{\int_{x_0}^{x} p(x')dx'}$. Therefore,

$$y = e^{-\int_{x_0}^{x} p(x')dx'}\left[\int_{x_0}^{x} q(x')e^{\int_{x_0}^{x'} p(x'')dx''}dx' + y_0\right].$$

Example 4. $(x^2+1)y' + 3xy = 6x$, $y(0) = 3$ is solved as $y(x) = 2 + (x^2+1)^{-3/2}$.

Homogeneous second-order linear equations

Consider $y'' + p(x)y' + q(x)y = 0$. The general solution is given by a superposition (linear combination) of two linearly independent solutions y_1, y_2: $y(x) = C_1y_1(x) + C_2y_2(x)$. ($C_1y_1 + C_2y_2 = 0$ for all x only when $C_1 = C_2 = 0$.)

Constant Coefficients

Consider $ay'' + by' + cy = 0$ ($a \neq 0$). By $y = e^{rx}$, we obtain $ar^2 + br + c = 0$. From this characteristic equation, we obtain two solutions $y_1(x) = e^{r_1x}$ and $y_2(x) = e^{r_2x}$, where $r_1, r_2 = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$, and the general solution is written as

$$y(x) = C_1e^{r_1x} + C_2e^{r_2x}.$$

If $ar^2 + br + c = 0$ has complex roots $\alpha \pm i\beta$, then the general solution can be written as

$$y(x) = C_1e^{(\alpha+i\beta)x} + C_2e^{(\alpha-i\beta)x} = e^{\alpha x}\left(c_1\cos\beta x + c_2\sin\beta x\right),$$

where $c_1 = C_1 + C_2$, $c_2 = i(C_1 - C_2)$.

If the characteristic equation has equal roots $r_1 = r_2 = r$, then two linearly independent solutions are found as $y_1 = e^{rx}$ and $y_2 = xe^{rx}$. Thus the general solution is given by

$$y(x) = C_1 y_1(x) + C_2 y_2(x) = (C_1 + C_2 x)\, e^{rx}.$$

Example 5. Let us solve $y'' + 2y' + y = 0$, $y(0) = 5$, $y'(0) = -3$. The characteristic equation is $r^2 + 2r + 1 = (r+1)^2 = 0$. Thus $r_1 = r_2 = -1$. The general solution is written as $y(x) = (C_1 + C_2 x)\, e^{-x}$. By the conditions at $x = 0$, we get $C_1 = 5$ and $C_2 = 2$. We obtain $y(x) = (5 + 2x)e^{-x}$.

Separation of variables [4]

Many linear PDEs can be reduced to linear ODEs with the method of separation of variables, described below.

We look for a separated solution (this is an ansatz[5])

$$u(x,y) = X(x)Y(y).$$

Consider Laplace's equation $u_{xx} + u_{yy} = 0$. We obtain

$$X''Y + XY'' = 0 \quad \Rightarrow \quad \frac{X''}{X} + \frac{Y''}{Y} = 0.$$

This implies X''/X and Y''/Y are constants.

Let λ be a constant and we write

$$X'' + \lambda X = 0, \quad Y'' - \lambda Y = 0.$$

We call λ the separation constant. At this moment λ is arbitrary. Thus the PDE reduced to two ODEs.

If $\lambda = 0$, then two ODEs have the following linearly independent solutions.

$$X = 1,\ x, \qquad Y = 1,\ y.$$

If $\lambda \neq 0$, then two ODEs have the following linearly independent solutions.

$$X = e^{\sqrt{-\lambda}x},\ e^{-\sqrt{-\lambda}x}, \qquad Y = e^{\sqrt{\lambda}y},\ e^{-\sqrt{\lambda}y}. \tag{1.2}$$

In either case, the solution is given by superpositions:

[4] This section corresponds to §0.2 of the textbook.

[5] an assumption or a guess to be verified later.

$$u = \begin{cases} (A_1 x + A_2)(B_1 y + B_2), & \lambda = 0, \\ \left(A_1 e^{\sqrt{-\lambda}x} + A_2 e^{-\sqrt{-\lambda}x}\right)\left(B_1 e^{\sqrt{\lambda}y} + B_2 e^{-\sqrt{\lambda}y}\right), & \lambda \neq 0, \end{cases} \tag{1.3}$$

where A_1, A_2, B_1, B_2 are constants. For $\lambda > 0$, by writing $\lambda = k^2$ ($k > 0$) we have

$$u(x,y) = \left(A_1 e^{ikx} + A_2 e^{-ikx}\right)\left(B_1 e^{ky} + B_2 e^{-ky}\right), \tag{1.4}$$

and for $\lambda < 0$, by writing $\lambda = -l^2$ ($l > 0$) we have

$$u(x,y) = \left(A_1 e^{lx} + A_2 e^{-lx}\right)\left(B_1 e^{ily} + B_2 e^{-ily}\right). \tag{1.5}$$

Instead of (1.2) we can also choose

$$X = \cos\left(\sqrt{\lambda}x\right),\ \sin\left(\sqrt{\lambda}x\right), \qquad Y = \cosh\left(\sqrt{\lambda}y\right),\ \sinh\left(\sqrt{\lambda}y\right).$$

In this case we have

$$u(x,y) = (A_1 \cos(kx) + A_2 \sin(kx))(B_1 \cosh(ky) + B_2 \sinh(ky)), \tag{1.6}$$

$$u(x,y) = (A_1 \cosh(lx) + A_2 \sinh(lx))(B_1 \cos(ly) + B_2 \sin(ly)). \tag{1.7}$$

Note that (1.6) becomes (1.4) and (1.7) becomes (1.5) by redefining the coefficients. We call solutions such as (1.3) through (1.7) separated solutions because they are given in the form $u(x,y) = X(x)Y(y)$.

The separation constant λ and coefficients A_1, A_2, B_1, B_2 are partially determined by boundary conditions. Suppose that our Laplace's equation is considered in the region $0 < x < L$, $0 < y < \infty$ with boundary conditions

$$u(0,y) = 0, \quad u(L,y) = 0, \quad u(x,0) = 0.$$

We find that $u = (A_1 x + A_2)(B_1 y + B_2)$ in (1.3) satisfies the boundary conditions only when $u = 0$.

We then find that (1.5) and (1.7) satisfy the conditions $u(0,y) = u(L,y) = 0$ only when $A_1 = A_2 = 0$. That is, only the solution $u = 0$ satisfies the boundary conditions.

Finally (1.4) and (1.6) satisfy $u(0,y) = u(L,y) = 0$ when $A_1 = 0$ and $k = n\pi/L$, where n is an integer. Furthermore we find $B_1 = 0$ by the condition $u(x,0) = 0$. That is, the solution $A_2 B_2 \sin(kx)\sinh(ky)$ with $k = n\pi/L$ ($n = 0, \pm 1, \pm 2, \ldots$) satisfies the boundary conditions.

Therefore we obtain the following separated solutions of Laplace's equation satisfying the boundary conditions.

$$u(x,y) = A \sin\frac{n\pi x}{L} \sinh\frac{n\pi y}{L}, \quad n = 1, 2, \ldots,$$

where A is a constant. Note that we still have infinitely many solutions. In general A depends on n. To have a unique solution, we need one more condition and need to use Fourier series.

Fourier series [6]

Let $A_0, A_1, B_1, \ldots$ be constants. The series below is called a trigonometric series.

$$A_0 + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right).$$

Suppose that a function $f(x) \in \mathbb{R}$, $-L < x < L$, is given by a trigonometric series:

$$f(x) = A_0 + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right). \tag{1.8}$$

Example 6. The function $f(x) = \cos(\pi x/L)$, $-L < x < L$, is given by the trigonometric series with $A_1 = 1$ and $A_0 = A_2 = A_3 = \cdots = B1 = B2 = \cdots = 0$.

For a given $f(x)$, we can determine the coefficients A_0, A_n, B_n making use of orthogonality relations.

Definition 1. The Kronecker delta δ_{mn} is defined as

$$\delta_{mn} = \begin{cases} 1, & m = n, \\ 0, & m \neq n. \end{cases}$$

Theorem 1 (Orthogonality relations). *Let n,m be integers. We assume $L > 0$. The following orthogonality relations hold.*

$$\int_{-L}^{L} \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx = \begin{cases} 2L & (n = m = 0), \\ L\delta_{nm} & (otherwise), \end{cases}$$

$$\int_{-L}^{L} \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = \begin{cases} 0 & (n = m = 0), \\ L\delta_{nm} & (otherwise), \end{cases}$$

$$\int_{-L}^{L} \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx = 0 \quad (all\, n,m).$$

Proof. If $n = m = 0$, then

[6] This section corresponds to §1.1 of the textbook.

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = \int_{-L}^{L} 1\cdot 1 dx = 2L.$$

Suppose that at least one of n, m is nonzero. We have

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = \frac{1}{2}\int_{-L}^{L}\left(\cos\frac{(n-m)\pi x}{L} + \cos\frac{(n+m)\pi x}{L}\right)dx$$

If $n \neq m$ nor $n \neq -m$, then

$$\begin{aligned}\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx &= \frac{1}{2}\left[\frac{L}{(n-m)\pi}\sin\frac{(n-m)\pi x}{L}\Big|_{-L}^{L} + \frac{L}{(n+m)\pi}\sin\frac{(n+m)\pi x}{L}\Big|_{-L}^{L}\right] \\ &= 0.\end{aligned}$$

If $n = \pm m$ ($m \neq 0$), then

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = \frac{1}{2}\int_{-L}^{L}\cos\frac{0\cdot\pi x}{L}dx = L.$$

Thus the orthogonality relations for $\int_{-L}^{L}\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx$ is proved. The orthogonality relations for $\int_{-L}^{L}\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx$ and $\int_{-L}^{L}\sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx$ are similarly proved. □

To determine A_0 in (1.8), we multiply $\cos\frac{0\cdot\pi x}{L} = 1$ on both sides and integrate with respect to x:

$$\int_{-L}^{L} f(x)dx = \int_{-L}^{L} A_0 dx + \sum_{n=1}^{\infty}\int_{-L}^{L}\left(A_n\cos\frac{n\pi x}{L} + B_n\sin\frac{n\pi x}{L}\right)dx = \int_{-L}^{L} A_0 dx = 2A_0.$$

To determine A_m ($m = 1, 2, \dots$) in (1.8), we multiply $\cos\frac{m\pi x}{L}$ on both sides and integrate with respect to x:

$$\begin{aligned}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}dx &= \int_{-L}^{L} A_0\cos\frac{m\pi x}{L}dx + \sum_{n=1}^{\infty}\int_{-L}^{L}\left(A_n\cos\frac{n\pi x}{L} + B_n\sin\frac{n\pi x}{L}\right)\cos\frac{m\pi x}{L}dx \\ &= \sum_{n=1}^{\infty} A_n\int_{-L}^{L}\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = \sum_{n=1}^{\infty} A_n L\delta_{nm} = LA_m.\end{aligned}$$

Similarly we can determine B_m ($m = 1, 2, \dots$) in (1.8) by multiplying $\sin\frac{m\pi x}{L}$ on both sides and integrate with respect to x:

$$\begin{aligned}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}dx &= \int_{-L}^{L} A_0\sin\frac{m\pi x}{L}dx + \sum_{n=1}^{\infty}\int_{-L}^{L}\left(A_n\cos\frac{n\pi x}{L} + B_n\sin\frac{n\pi x}{L}\right)\sin\frac{m\pi x}{L}dx \\ &= \sum_{n=1}^{\infty} B_n\int_{-L}^{L}\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx = \sum_{n=1}^{\infty} B_n L\delta_{nm} = LB_m.\end{aligned}$$

Therefore the Fourier coefficients in (1.8) are given by

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx, \quad A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx. \tag{1.9}$$

In general, for a given function $f(x) \in \mathbb{R}$, $-L < x < L$, the trigonometric series with coefficients (1.9) is called the Fourier series of f.

Example 7. Let us calculate the Fourier series of $f(x) = x$, $-L < x < L$. We have

$$A_0 = \frac{1}{2L}\int_{-L}^{L} xdx = 0,$$
$$A_n = \frac{1}{L}\int_{-L}^{L} x\cos\frac{n\pi x}{L}dx = \frac{1}{L}\left[\frac{L}{n\pi}x\sin\frac{n\pi x}{L}\bigg|_{-L}^{L} - \frac{L}{n\pi}\int_{-L}^{L}\sin\frac{n\pi x}{L}dx\right] = 0,$$
$$B_n = \frac{1}{L}\int_{-L}^{L} x\sin\frac{n\pi x}{L}dx = \frac{1}{L}\left[-\frac{L}{n\pi}x\cos\frac{n\pi x}{L}\bigg|_{-L}^{L} + \frac{L}{n\pi}\int_{-L}^{L}\cos\frac{n\pi x}{L}dx\right] = \frac{2L}{n\pi}(-1)^{n+1}.$$

Therefore we obtain

$$x = \sum_{n=1}^{\infty}\frac{2L}{n\pi}(-1)^{n+1}\sin\frac{n\pi x}{L}, \qquad -L < x < L.$$

Even and odd functions

A function $f(x)$ is even if $f(-x) = f(x)$ and is odd if $f(-x) = -f(x)$.

Example 8. x is odd and x^2 is even. But $x + x^2$ is neither.

Example 9. $\cos x$ is even and $\sin x$ is odd. For constants a, b, $\sin(ax)\cos(bx)$ is odd because $\sin(-ax)\cos(-bx) = [-\sin(ax)][\cos(bx)] = -\sin(ax)\cos(bx)$. But $\sin(ax)\sin(bx)$ is even because $\sin(-ax)\sin(-bx) = [-\sin(ax)][-\sin(bx)] = \sin(ax)\sin(bx)$.

Note that

$$\int_{-L}^{L} f(x)dx = \begin{cases} 2\int_0^L f(x)dx & \text{(even function)}, \\ 0 & \text{(odd function)}. \end{cases}$$

If $f(x)$ is even, then

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{L}.$$

If $f(x)$ is odd, then

$$f(x) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{L}.$$

Example 10. Let us compute the Fourier series of $f(x) = x$, $-L < x < L$. Since f is odd, $A_0, A_1, \ldots$ are zero. We have

$$\begin{aligned} B_n &= \frac{1}{L}\int_{-L}^{L} x \sin\frac{n\pi x}{L} dx = \frac{2}{L}\int_0^L x\sin\frac{n\pi x}{L} dx \\ &= \frac{2}{L}\left[-x\frac{L}{n\pi}\cos\frac{n\pi x}{L}\Big|_0^L + \frac{L}{n\pi}\int_0^L \cos\frac{n\pi x}{L} dx\right] = \frac{2}{L}\left[-\frac{L^2}{n\pi}\cos(n\pi)\right] \\ &= -\frac{2L}{n\pi}\cos n\pi = \frac{2L}{n\pi}(-1)^{n+1}. \end{aligned}$$

Therefore,

$$x = \frac{2L}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\frac{n\pi x}{L}, \qquad -L < x < L. \tag{1.10}$$

Note that each term on the right-hand side is zero when $x = \pm L$. Because of this, the above sum does not converge uniformly to x and there appear oscillations near $-L$ and L in Fig. 1.1. This is known as the Gibbs phenomenon [7], which shows up when the function has discontinuities.

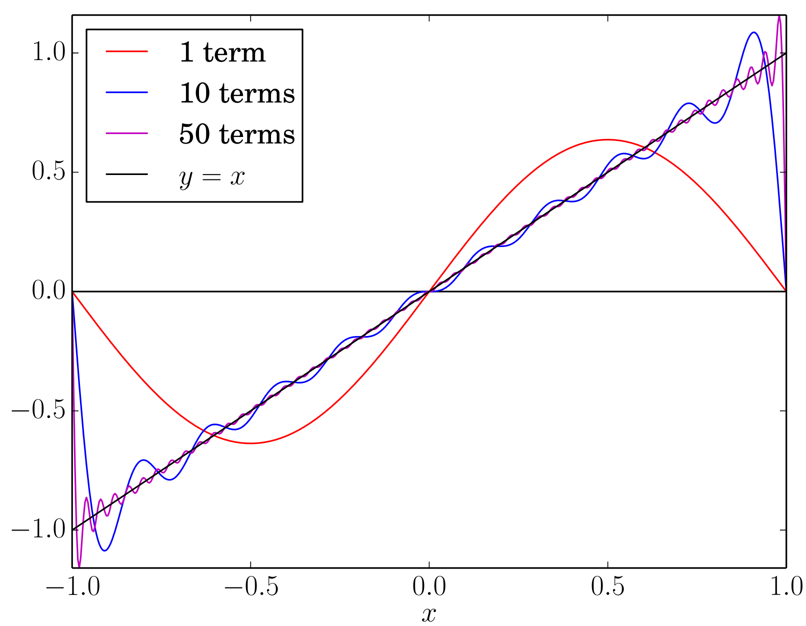

Fig. 1.1 Example 10. $x = \frac{2L}{\pi}\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{L}$, $-L < x < L$, where $L = 1$, and $N = 1, 10, 50$.

[7] See §1.3 of the textbook.

The following formulas are useful.

$$\int_{-L}^{L} x^k \sin\frac{n\pi x}{L}dx = \left[-\frac{Lx^k}{n\pi}\cos\frac{n\pi x}{L}\right]_{-L}^{L} + \frac{Lk}{n\pi}\int_{-L}^{L} x^{k-1}\cos\frac{n\pi x}{L}dx,$$
$$\int_{-L}^{L} x^k \cos\frac{n\pi x}{L}dx = \left[\frac{Lx^k}{n\pi}\sin\frac{n\pi x}{L}\right]_{-L}^{L} - \frac{Lk}{n\pi}\int_{-L}^{L} x^{k-1}\sin\frac{n\pi x}{L}dx.$$

Periodic functions

A function $f(x)$, $-\infty < x < \infty$, is $2L$-periodic if $f(x+2L) = f(x)$ for $x \in (-\infty,\infty)$. Note that $\sin(n\pi x/L)$ and $\cos(n\pi x/L)$ are $2L$-periodic:

$$\sin\frac{n\pi(x+2L)}{L} = \sin\left(\frac{n\pi x}{L} + 2n\pi\right) = \sin\frac{n\pi x}{L},$$
$$\cos\frac{n\pi(x+2L)}{L} = \cos\left(\frac{n\pi x}{L} + 2n\pi\right) = \cos\frac{n\pi x}{L}.$$

Therefore any convergent trigonometric series

$$A_0 + \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi x}{L} + B_n\sin\frac{n\pi x}{L}\right)$$

defines a $2L$-periodic function on $-\infty < x < \infty$.

We divide $(-\infty,\infty)$ into intervals $(2n-1)L < x < (2n+1)L$ $(n = 0,\pm1,\pm2,\dots)$ and we can focus on one of these intervals. In particular we can restrict x to $-L < x < L$.

Example 11. Let us consider the Fourier series of the $2L$-periodic function $f(x)$,

$$f(x) = \begin{cases} -1, & (2n-1)L < x < 2nL, \\ 1, & 2nL < x < (2n+1)L, \end{cases}$$

where $n = 0,\pm1,\dots$. In particular $f(x) = -1$ for $-L < x < 0$ and $f(x) = 1$ for $0 < x < L$. Thus f is an odd function. The Fourier series is given by

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin\frac{n\pi x}{L}, \qquad -\infty < x < \infty.$$

Fourier sine and cosine series

Consider the Fourier series of $f(x)$, $0 < x < L$. Since the orthogonality relations are given for the interval $-L < x < L$, we need to extend f. There are two ways to extend the function. That is, the Fourier sine series and the Fourier cosine series.

Fourier sine series

The first way is to define the odd extension $f_O(x)$ as

$$f_O(x) = \begin{cases} f(x), & 0 < x < L, \\ 0, & x = 0, \\ -f(-x), & -L < x < 0. \end{cases}$$

We note that $f_O(x)$ is odd. The Fourier series is given by

$$f_O(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}, \quad -L < x < L,$$

where

$$B_n = \frac{1}{L} \int_{-L}^{L} f_O(x) \sin \frac{n\pi x}{L} dx = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx.$$

On the interval $0 < x < L$ we have

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}, \quad 0 < x < L, \tag{1.11}$$

where

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx. \tag{1.12}$$

This series is called the Fourier sine series.

Example 12. The Fourier sine series of $f(x) = x$, $0 < x < L$, is obtained through the extension $f_O(x)$. In this case $f_O(x) = x$ and we obtain the series (1.10) on $0 < x < L$.

Fourier cosine series

The second way is to define the even extension $f_E(x)$ as

$$f_E(x) = \begin{cases} f(x), & 0 < x < L, \\ 0, & x = 0, \\ f(-x), & -L < x < 0. \end{cases}$$

We note that $f_E(x)$ is even. Indeed the value $f_E(0)$ is arbitrary and not necessarily zero. The Fourier series is given by

$$f_E(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}, \quad -L < x < L,$$

where

$$\begin{aligned} A_0 &= \frac{1}{2L}\int_{-L}^{L} f_E(x)dx = \frac{1}{L}\int_0^L f(x)dx, \\ A_n &= \frac{1}{2L}\int_{-L}^{L} f_E(x)\cos\frac{n\pi x}{L}dx = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx. \end{aligned}$$

On the interval $0 < x < L$ we have

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}, \quad 0 < x < L, \tag{1.13}$$

where

$$A_0 = \frac{1}{L}\int_0^L f(x)dx, \quad A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx. \tag{1.14}$$

This series is called the Fourier cosine series.

Thus a function $f(x)$, $0 < x < L$, is expressed either in the Fourier sine series (1.11), (1.12), or the Fourier cosine series (1.13), (1.14). Figures 1.1 and 1.2 show that the convergence rates of these two series are generally different.

Example 13. Let us obtain the Fourier cosine series of $f(x) = x$, $0 < x < L$. We extend f as

$$f_E(x) = \begin{cases} x, & 0 < x < L, \\ 0, & x = 0, \\ -x, & -L < x < 0. \end{cases}$$

Indeed $f_E(x) = |x|$.

$$\begin{aligned} A_0 &= \frac{1}{2L}\int_{-L}^{L} f_E(x)dx = \frac{1}{L}\int_0^L xdx = \frac{L}{2}, \\ A_n &= \frac{1}{2L}\int_{-L}^{L} f_E(x)\cos\frac{n\pi x}{L}dx = \frac{2}{L}\int_0^L x\cos\frac{n\pi x}{L}dx \\ &= \frac{2}{L}\left[\frac{L}{n\pi}x\sin\frac{n\pi x}{L}\Big|_0^L - \frac{L}{n\pi}\int_0^L \sin\frac{n\pi x}{L}dx\right] = \frac{2L}{(n\pi)^2}\left((-1)^n - 1\right). \end{aligned}$$

Therefore,

$$x = \frac{L}{2} + \frac{2L}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2}\cos\frac{n\pi x}{L}, \qquad 0 < x < L. \tag{1.15}$$

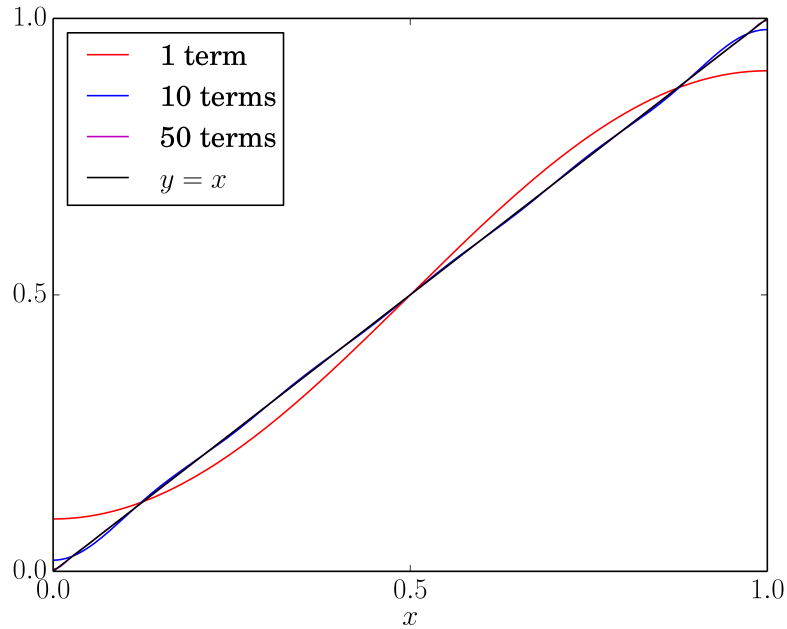

Fig. 1.2 Example 13. $x = \frac{L}{2} + \frac{2L}{\pi^2}\sum_{n=1}^{N} \frac{(-1)^n - 1}{n^2} \cos\frac{n\pi x}{L}$, $0 < x < L$, where $L = 1$, and $N = 1, 10, 50$.

Convergence of Fourier series [8]

Definition 2. For a given $f(x)$, let us write

$$f(x+0) = \lim_{\varepsilon \to 0} f(x+\varepsilon), \quad f(x-0) = \lim_{\varepsilon \to 0} f(x-\varepsilon),$$

where $\varepsilon > 0$.

Definition 3 (Piecewise continuous). A function $f(x)$, $a < x < b$, is said to be piecewise continuous if there is a finite set of points $a = x_0 < x_1 < \cdots < x_p < x_{p+1} = b$ such that $f(x)$ is continuous at $x \neq x_i$ $(i = 1, \ldots, p)$, $f(x_i + 0)$ $(i = 0, \ldots, p)$ exists, and $f(x_i - 0)$ $(i = 1, \ldots, p+1)$ exists.

Definition 4 (Piecewise smooth). A function $f(x)$, $a < x < b$, is said to be piecewise smooth if $f(x)$ and all of its derivatives are piecewise continuous.

Example 14. The function $f(x) = |x|$, $-L < x < L$, is piecewise smooth. The function $f(x) = x^2 \sin(1/x)$, $-L < x < L$, is piecewise continuous but is not piecewise smooth because $\lim_{\varepsilon \to 0} f'(0 \pm \varepsilon)$ does not exist. The function $f(x) = 1/(x^2 - L^2)$,

[8] This section corresponds to §1.2 of the textbook.

$-L < x < L$, is not piecewise continuous because $f(-L+0)$ and $f(L-0)$ are not finite.

Theorem 2 (Convergence theorem). *Let $f(x)$, $-L < x < L$, be piecewise smooth. Then the Fourier series of f converges for all x to the value $\frac{1}{2}\left[\bar{f}(x+0)+\bar{f}(x-0)\right]$, where $\bar{f}$ is the $2L$-periodic extension of f.*

If $f(x)$ is continuous on $[-L,L]$ and $f(-L) = f(L)$ in addition to the conditions assumed in the above theorem, then the Fourier series uniformly converges. For example, the Fourier series of $f(x) = |x|$, $-L < x < L$, (see (1.15)) uniformly converges.

Parseval's Theorem and Mean Square Error [9]

Theorem 3 (Parseval's theorem). *Let $f(x)$, $-L < x < L$, be a piecewise smooth function with Fourier series $A_0 + \sum_{n=1}^{\infty}\left[A_n\cos(n\pi x/L)+B_n\sin(n\pi x/L)\right]$. Then,*

$$\text{mean square of } f(x) = \frac{1}{2L}\int_{-L}^{L} f(x)^2 dx = A_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(A_n^2+B_n^2\right).$$

Proof. Direct calculation of the integral using the Fourier series of f. □

We define the mean square error σ_N^2 as

$$\sigma_N^2 = \frac{1}{2L}\int_{-L}^{L}\left[f(x)-f_N(x)\right]^2 dx,$$

where

$$f_N(x) = A_0 + \sum_{n=1}^{N}\left[A_n\cos(n\pi x/L)+B_n\sin(n\pi x/L)\right].$$

By Parseval's theorem, we obtain

$$\sigma_N^2 = \frac{1}{2}\sum_{n=N+1}^{\infty}\left(A_n^2+B_n^2\right).$$

[9] This section corresponds to §1.4 of the textbook.

Example 15. Let us find σ_N^2 for $f(x)=x$, $-L<x<L$. From Example 10, we have $A_0=A_n=0$ and

$$f_N(x)=\sum_{n=1}^{N}B_n\sin\frac{n\pi x}{L},\quad B_n=\frac{2L}{n\pi}(-1)^{n+1}.$$

This is also the Fourier sine series of x, $0<x<L$, in Example 12. We obtain

$$\sigma_N^2=\frac{1}{2}\sum_{n=N+1}^{\infty}\left(\frac{2L}{n\pi}(-1)^{n+1}\right)^2=\frac{2L^2}{\pi^2}\sum_{n=N+1}^{\infty}\frac{1}{n^2}.$$

Note that

$$\int_{N+1}^{\infty}\frac{1}{x^2}dx=\int_{N}^{\infty}\frac{1}{(x+1)^2}dx\le\sum_{n=N+1}^{\infty}\frac{1}{n^2}\le\int_{N+1}^{\infty}\frac{1}{(x-1)^2}dx=\int_{N}^{\infty}\frac{1}{x^2}dx.$$

We have

$$\int_{N}^{\infty}\frac{1}{(x+1)^2}dx=\frac{1}{N+1}=\frac{1}{N}\left(1-\frac{1}{N}+\frac{1}{N^2}-\frac{1}{N^3}+\cdots\right),\qquad\int_{N}^{\infty}\frac{1}{x^2}dx=\frac{1}{N}.$$

Let us introduce the symbol O (this is called "big O") to express the order. For some f_N, $f_N=O\left(N^{-1}\right)$ as $N\to\infty$ means that there exist a constant $C>0$ and a number N_0 such that $|f_N|\le CN^{-1}$ for all $N>N_0$. Therefore we obtain

$$\sigma_N^2=\frac{2L^2}{\pi^2}\frac{1}{N}\left[1+O\left(\frac{1}{N}\right)\right]=O\left(N^{-1}\right),\quad N\to\infty.\tag{1.16}$$

We note that σ_N^2 goes to zero as $N\to\infty$ although we know that the sum in (1.10) does not converge uniformly. This happened because we considered the mean square and took the integral.

Note that each term on the right-hand side is zero when $x=\pm L$. Because of this, the above sum to x and there appear oscillations near $-L$ and L in Fig. 1.1.

Example 16. Let us find σ_{2N}^2 for $f(x)=|x|$, $-L<x<L$. From Example 13 we know that $B_n=0$, $A_{2m}=0$ ($m=1,2,\dots$), and

$$f_{2N}(x)=A_0+\sum_{m=1}^{N}A_{2m-1}\cos\frac{(2m-1)\pi x}{L},\quad A_0=\frac{L}{2},\quad A_{2m-1}=-\frac{4L}{\pi^2(2m-1)^2}.$$

This is also the Fourier cosine series of x, $0<x<L$, in Example 13. Hence we obtain

$$\sigma_{2N}^2=\frac{1}{2}\sum_{n=2N+1}^{\infty}A_n^2=\frac{1}{2}\sum_{m=N+1}^{\infty}A_{2m-1}^2=\frac{8L^2}{\pi^4}\sum_{m=N+1}^{\infty}\frac{1}{(2m-1)^4}.$$

Note that

$$\int_N^\infty \frac{1}{(2x+1)^4}dx \le \sum_{m=N+1}^\infty \frac{1}{(2m-1)^4} \le \int_N^\infty \frac{1}{(2x-1)^4}dx,$$

and LHS $= \frac{1}{6(2N+1)^3} = \frac{1}{48N^3} + O\left(N^{-4}\right)$ and RHS $= \frac{1}{6(2N-1)^3} = \frac{1}{48N^3} + O\left(N^{-4}\right)$. Therefore we obtain

$$\sigma_{2N}^2 = \frac{L^2}{6\pi^4 N^3} + O\left(N^{-4}\right) = O\left(N^{-3}\right), \quad N \to \infty. \tag{1.17}$$

Thus the Fourier series of x converges as $O(1/N)$ and the Fourier series of $|x|$ converges as $O(1/N^3)$. Equations (1.16) and (1.17) explain the difference between Figs. 1.1 and 1.2.

Complex form of Fourier series [10]

Throughout this course, we use a bar to indicate complex conjugate. That is, if $c = a + ib$, then $\overline{c} = a - ib$. Suppose $f(x) \in \mathbb{R}$ is given. Using Euler's formula (1.1), we can rewrite the Fourier series of f as follows.

$$f(x) = A_0 + \sum_{n=1}^\infty \left(A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}\right) = A_0 + \sum_{n=1}^\infty \left(\frac{A_n - iB_n}{2}e^{in\pi x/L} + \frac{A_n + iB_n}{2}e^{-in\pi x/L}\right).$$

We define

$$\alpha_0 = A_0, \quad \alpha_n = \frac{A_n - iB_n}{2}, \quad \alpha_{-n} = \frac{A_n + iB_n}{2}.$$

We obtain

$$f(x) = \alpha_0 + \sum_{n=1}^\infty \left(\alpha_n e^{in\pi x/L} + \alpha_{-n}e^{-in\pi x/L}\right) = \sum_{n=-\infty}^\infty \alpha_n e^{in\pi x/L}. \tag{1.18}$$

This is the Fourier series in complex form. Using (1.9), α_n ($n = 0, \pm 1, \pm 2, \dots$) are given by

$$\alpha_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}dx. \tag{1.19}$$

We have the following orthogonality relations.

$$\int_{-L}^{L} e^{in\pi x/L}e^{-im\pi x/L}dx = \int_{-L}^{L} e^{i(n-m)\pi x/L}dx = 2L\delta_{mn}.$$

[10] This section corresponds to §1.5 of the textbook.

With the help of the orthogonality relations, we can directly obtain (1.19) by integrating both sides of (1.18):

$$\begin{aligned}\int_{-L}^{L} f(x)e^{-in\pi x/L}dx &= \int_{-L}^{L}\left(\sum_{n'=-\infty}^{\infty}\alpha_{n'}e^{in'\pi x/L}\right)e^{-in\pi x/L}dx \\ &= \sum_{n'=-\infty}^{\infty}\alpha_{n'}\int_{-L}^{L}e^{i(n'-n)\pi x/L}dx = \sum_{n'=-\infty}^{\infty}\alpha_{n'}2L\delta_{nn'} = 2L\alpha_n.\end{aligned}$$

We can write Parseval's theorem as follows in complex form.

$$\frac{1}{2L}\int_{-L}^{L} f(x)^2dx = \sum_{n=-\infty}^{\infty}|\alpha_n|^2.$$

This is seen by the calculation below.

$$\begin{aligned}\frac{1}{2L}\int_{-L}^{L} f(x)^2dx &= A_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(A_n^2 + B_n^2\right) \\ &= A_0^2 + 2\sum_{n=1}^{\infty}\frac{A_n - iB_n}{2}\frac{A_n + iB_n}{2} \\ &= \alpha_0^2 + 2\sum_{n=1}^{\infty}\alpha_n\alpha_{-n} = \alpha_0^2 + \sum_{n=1}^{\infty}|\alpha_n|^2 + \sum_{n=1}^{\infty}|\alpha_{-n}|^2 \\ &= \alpha_0^2 + \sum_{n=1}^{\infty}|\alpha_n|^2 + \sum_{n=-\infty}^{-1}|\alpha_n|^2 \\ &= \sum_{n=-\infty}^{\infty}|\alpha_n|^2.\end{aligned}$$