Problem Set 9 (3/19, 3/21, 3/26)
Due on Fri, Apr 4

1) Find the general solution of $\nabla^2[f(r)] = 0$.
   Solution: $f(r) = A + B/r$, where A, B are arbitrary constants.

2) Solve $\nabla^2[f(r)] = -1$ with the boundary condition $f(a) = 0$ and $f(0)$ finite.
   Solution: $f(r) = (a^2 - r^2)/6$.

3) Find the solution $u(r,t)$ of the heat equation $u_t = K\nabla^2 u$, $-\infty < t < \infty$, in the sphere $0 \le r < a$ satisfying the boundary condition $u(a,t) = 3\cos 2t$.
   Solution: We use $u(a,t) = 3e^{2it}$ and take the real part in the end. $u(r,t) = \frac{3a}{r} \operatorname{Re}\left[e^{c(r-a)(1+i)} e^{2it} \frac{1-e^{-2cr(1+i)}}{1-e^{-2ca(1+i)}}\right]$, where $c = 1/\sqrt{K}$.

4) (a) Compute $P_1'(0)$, $P_2'(0)$, $P_3'(0)$, $P_4'(0)$ by differentiating Legendre polynomials.
   (b) Find $P_{2n+1}'(0)$ and $P_{2n}'(0)$ $(n = 0, 1, 2, \cdots)$ using Rodrigues' formula.
   Solution: (a) $P_1'(0) = 1$, $P_2'(0) = 0$, $P_3'(0) = -3/2$, $P_4'(0) = 0$. (b) $P_{2n+1}'(0) = (-1)^n \frac{(2n+2)!}{n!(n+1)!2^{2n+1}}$ and $P_{2n}'(0) = 0$.

5) Let $f(s) = 0$ for $-1 < s < 0$ and $f(s) = 1$ for $0 < s < 1$. Find the expansion of $f(s)$ in a series of Legendre polynomials.
   Solution: $\frac{1}{2} + \sum_{k=1}^{\infty} \frac{2k+1}{2k(k+1)} P_k'(0) P_k(s) = \frac{1}{2}[f(s-0) + f(s+0)]$, $-1 < s < 1$.

6) Find the solution of Laplace's equation $\nabla^2 u = 0$ in the sphere $0 \le r < a$ satisfying the boundary condition $u(a,\theta) = 1$ if $0 < \theta < \pi/2$ and $u(a,\theta) = 0$ otherwise.
   Solution: $u(r,\theta) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{(2k+1)P_k'(0)}{2k(k+1)} \left(\frac{r}{a}\right)^k P_k(\cos\theta)$.