Problem Set 1 (1/8, 1/10, 1/13)
Due on Fri, Jan 18

1) Classify each of the following second-order equations as elliptic, parabolic, or hyperbolic.
   (a) $u_{xx} + 3u_{xy} + u_{yy} + 2u_x - u_y = 0$,
   (b) $u_{xx} + 3u_{xy} + 8u_{yy} + 2u_x - u_y = 0$,
   (c) $u_{xx} - 2u_{xy} + u_{yy} + 2u_x - u_y = 0$,
   (d) $u_{xx} + xu_{yy} = 0$.

   Solution: (a) hyperbolic, (b) elliptic, (c) parabolic, (d) elliptic if $x > 0$ and hyperbolic if $x < 0$.

2) Prove the formulas $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$ and $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$.
   Solution: $\sinh x \cosh y + \cosh x \sinh y = \frac{e^x - e^{-x}}{2}\frac{e^y + e^{-y}}{2} + \frac{e^x + e^{-x}}{2}\frac{e^y - e^{-y}}{2} = \sinh(x+y)$.
   $\cosh x \cosh y + \sinh x \sinh y = \frac{e^x + e^{-x}}{2}\frac{e^y + e^{-y}}{2} + \frac{e^x - e^{-x}}{2}\frac{e^y - e^{-y}}{2} = \cosh(x + y)$.

3) Find the general solutions.
   (a) $y' = ky(1 - y)$,
   (b) $xy' + 4y = x^2$,
   (c) $y'' + 4y' + 4y = 0$,
   (d) $y'' + 2y' - 15y = 0$.

   Solution: (a) $y(x) = 1/\left(1 + Ce^{-kx}\right)$, (b) $y(x) = x^2/6 + C/x^4$, (c) $y(x) = C_1e^{-2x} + C_2xe^{-2x}$, and (d) $y(x) = C_1e^{3x} + C_2e^{-5x}$.

4) Find the separated equations satisfied by $X(x)$, $Y(y)$ for the following PDEs.
   (a) $u_{xx} - 2u_{yy} = 0$,
   (b) $u_{xx} + u_{yy} + 2u_x = 0$,
   (c) $x^2u_{xx} - 2yu_y = 0$,
   (d) $u_{xx} + u_x + u_y - u = 0$.

   Solution: (a) $X'' - 2\lambda X = 0$, $Y'' - \lambda Y = 0$, (b) $X'' + 2X' + \lambda X = 0$, $Y'' - \lambda Y = 0$, (c) $x^2X'' - \lambda X = 0$, $2yY' - \lambda Y = 0$, (d) $X'' + X' - \lambda X = 0$, $Y' + (\lambda - 1)Y = 0$.

5) Find the separated solutions of $u_{xx} + yu_y + u = 0$.
   Solution:

$$u(x, y) = \begin{cases} \left(A_1e^{x\sqrt{\lambda}} + A_2e^{-x\sqrt{\lambda}}\right)\left(1/|y|^{1+\lambda}\right) & \text{for } \lambda > 0, \\ (A_1 + A_2x)\left(1/|y|\right) & \text{for } \lambda = 0, \\ \left(A_1 \cos x\sqrt{-\lambda} + A_2 \sin x\sqrt{-\lambda}\right)\left(1/|y|^{1+\lambda}\right) & \text{for } \lambda < 0. \end{cases}$$

6) Find the separated solutions $u(x, y)$ of Laplace's equation $u_{xx}+u_{yy} = 0$ in the region $0 < x < L$, $y > 0$, that satisfy the boundary conditions $u(0, y) = 0$, $u(L, y) = 0$ and the boundedness condition $|u(x, y)| \leq M$ for $y > 0$, where M is a constant independent of (x, y).
Solution:

$$u(x, y) = C \sin \frac{n\pi x}{L} e^{-n\pi y/L} \quad (n = 1, 2, \ldots).$$

7) Find the separated solutions $u(x, t)$ of the heat equation $u_t - u_{xx} = 0$ in the region $0 < x < L$, $t > 0$, that satisfy the boundary conditions $u(0, t) = 0$, $u(L, t) = 0$.
Solution:

$$u(x,t) = \begin{cases} \left(A_1 e^{kx} + A_2 e^{-kx}\right) e^{k^2 t} & \text{for } \lambda = k^2,\ k > 0, \\ A_1 x + A_2 & \text{for } \lambda = 0, \\ \left(A_1 e^{ilx} + A_2 e^{-ilx}\right) e^{-l^2 t} & \text{for } \lambda = -l^2,\ l > 0. \end{cases}$$

The case $\lambda < 0$ satisfies the boundary conditions. We obtain

$$u(x, t) = C \sin \frac{n\pi x}{L} e^{-(n\pi/L)^2 t} \quad (n = 1, 2, \ldots).$$