Matrix Project
Due on Fri, Jul 18

Let us numerically solve $A\vec{x} = \vec{b}$ with Gaussian-Jordan elimination. Here A is an $n \times n$ matrix and $\vec{b}$ is a vector with n components. We will write a numerical code to obtain $\vec{x}$.

To consider the algorithm, we begin with the following example. Suppose we have a system of equations

$$\begin{cases} 2x_1 - x_2 \phantom{{}-x_3} &= 1, \\ -x_1 + 2x_2 - x_3 &= 0, \\ -x_2 \phantom{{}+2x_2} + 2x_3 &= 1. \end{cases}$$

We write the augmented matrix as

$$\left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right] = \left[\begin{array}{rrr:r} 2 & -1 & 0 & 1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & 1 \end{array}\right].$$

By (2nd row) $- \frac{a_{21}}{a_{11}}$(1st row), we obtain

$$\left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22} - m_{21}a_{12} & a_{23} - m_{21}a_{13} & b_2 - m_{21}b_1 \\ 0 & a_{32} & a_{33} & b_3 \end{array}\right] = \left[\begin{array}{ccc:c} 2 & -1 & 0 & 1 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} \\ 0 & -1 & 2 & 1 \end{array}\right],$$

where $m_{21} = a_{21}/a_{11} = -1/2$. Now we have

$$\left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22} & a_{23} & b_2 \\ 0 & a_{32} & a_{33} & b_3 \end{array}\right] = \left[\begin{array}{ccc:c} 2 & -1 & 0 & 1 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} \\ 0 & -1 & 2 & 1 \end{array}\right].$$

By (3rd row) $- m_{32}$(2nd row), where $m_{32} = a_{32}/a_{22} = -1/\frac{3}{2} = -2/3$, we obtain

$$\left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22} & a_{23} & b_2 \\ 0 & a_{32} - m_{32}a_{22} & a_{33} - m_{32}a_{23} & b_3 - m_{32}b_2 \end{array}\right] = \left[\begin{array}{ccc:c} 2 & -1 & 0 & 1 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} \\ 0 & 0 & \frac{4}{3} & \frac{4}{3} \end{array}\right].$$

Thus we obtain the upper triangular matrix

$$\left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22} & a_{23} & b_2 \\ 0 & 0 & a_{33} & b_3 \end{array}\right] = \left[\begin{array}{ccc:c} 2 & -1 & 0 & 1 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} \\ 0 & 0 & \frac{4}{3} & \frac{4}{3} \end{array}\right],$$

or the system

$$\begin{cases} 2x_1 - \quad x_2 \qquad\quad = 1, \\ \qquad \frac{3}{2}x_2 - \quad x_3 = \frac{1}{2}, \\ \qquad\qquad\quad \frac{4}{3}x_3 = \frac{4}{3}. \end{cases}$$

The matrix is not in reduced row-echelon form but we will find below that this triangular matrix is enough.

We implement Gauss-Jordan elimination by generalizing the above example. An upper triangular matrix (row-echelon form) is obtained as follows.

```
for k=1:n-1   % k: step index
for i=k+1:n
  m(i,k)=a(i,k)/a(k,k)
   for j=k+1:n
     a(i,j)=a(i,j)-m(i,k)*a(k,j)
   end
   b(i)=b(i)-m(i,k)*b(k)
end
```

We note that $a_{kk} \neq 0$ is assumed[1].

Now A is an upper triangular matrix. In general we have

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ \qquad\quad a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ & \vdots \\ \qquad\qquad\qquad\qquad\quad a_{nn}x_n &= b_n \end{cases}$$

[1] Although we assume $a_{kk} \neq 0$ for our project, sometimes $a_{kk} = 0$ or a_{kk} becomes very close to 0. If $a_{kk} = 0$, then we can find index l such that $|a_{lk}| = \max\{|a_{ik}| : k \leq i \leq n\}$, interchange the lth row and kth row, and proceed with the elimination. If A is invertible, then Gauss-Jordan elimination with this swap process (partial pivoting) does not break down (Math 571).

We can readily obtain $x_n, x_{n-1}, \ldots, x_1$ by back substitution:

$$
\begin{aligned}
x_n &= \frac{b_n}{a_{nn}}, \\
x_{n-1} &= \frac{b_{n-1} - a_{n-1,n}x_n}{a_{n-1,n-1}}, \\
&\vdots \\
x_1 &= \frac{b_1 - (a_{12}x_2 + \cdots + a_{1n}x_n)}{a_{11}}.
\end{aligned}
$$

This procedure can be implemented as follows.

```
x(n)=b(n)/a(n,n)
for i=n-1:-1:1   % i: row index
  tmp=b(i)
  for j=i+1:n    % j: column index
    tmp=tmp-a(i,j)*x(j)
  end
  x(i)=tmp/a(i,i)
end
```

In this way we can solve $A\vec{x} = \vec{b}$ and obtain the solution vector $\vec{x}$.