A remark on the generalized convexity condition and propagation of boundary-induced discontinuity in stationary radiative transfer*

Daisuke Kawagoe †

We consider the boundary value problem of the stationary transport equation (STE):

$$\begin{cases} \xi \cdot \nabla_x f(x,\xi) + \mu_t(x) f(x,\xi) = \mu_s(x) \int_{S^{d-1}} p(x,\xi,\xi') f(x,\xi')\, d\sigma_{\xi'} & \text{in } \Omega \times S^{d-1}, \\ f(x,\xi) = f_0(x,\xi) & \text{on } \Gamma_-. \end{cases} \tag{1}$$

Here, Ω is a bounded convex domain in $\mathbb{R}^d$ $(d = 2,3)$ with C^1 boundary $\partial\Omega$, S^{d-1} is the unit sphere in $\mathbb{R}^d$, and "boundaries" $\Gamma_\pm$ are defined by

$$\Gamma_\pm := \{(x,\xi) \in \partial\Omega \times S^{d-1} \mid \pm n(x) \cdot \xi > 0\},$$

where $n(x)$ is the outer unit normal vector at $x \in \partial\Omega$. STE is a mathematical model describing propagation of photon in turbid media, especially in human bodies [1].

In [3], we described the boundary-induced discontinuity, which is discontinuity of the solution arising from the discontinuous boundary data f_0, and proposed a way of reconstructing the attenuation coefficient μ_t by measuring the jump of the solution at discontinuous points on the boundary Γ_+. In this result, we assumed the generalized convexity condition on the domain.

The generalized convexity condition is described as follows: Let Ω be a bounded convex domain in $\mathbb{R}^d$ with the C^1 boundary $\partial\Omega$. We assume that $\overline{\Omega} = \cup_{j=1}^N \overline{\Omega_j}$, where Ω_j, $1 \leq j \leq N$, are disjoint (open) subdomains of Ω with piecewise C^1 boundaries. Let $\Omega_0 := \cup_{j=1}^N \Omega_j$. We assume that, for all $(x,\xi) \in \Omega \times S^{d-1}$, the half line $\{x - t\xi \mid t \geq 0\}$ intersects with $\partial\Omega_0$ at most finite times. In other words, for all $(x,\xi) \in \Omega \times S^{d-1}$, there exist positive integer $l(x,\xi)$ and real numbers $\{t_j(x,\xi)\}_{j=1}^{l(x,\xi)}$ such that $0 \leq t_1(x,\xi) < t_2(x,\xi) < \cdots < t_{l(x,\xi)}(x,\xi)$, $x - t\xi \in \partial\Omega_0$ if and only if $t = t_j(x,\xi)$, and $\sup_{(x,\xi)\in\Omega\times S^{d-1}} l(x,\xi) < \infty$. In [3], we assumed that the coefficients μ_s, μ_t and the integral kernel p are bounded continuous in Ω_0 and may be discontinuous on $\partial\Omega_0$.

In this talk, we will discuss discontinuity of the solution to (1) when the boundary $\partial\Omega_0$ has a flat part, in which the generalized convexity condition is violated. In particular, we will see that effect of the flat parts is negligible in the reconstruction process, though location of discontinuous points of the solution becomes not so explicit.

References

[1] S. R. Arridge, J. C. Schotland, Optical tomography: forward and inverse problems, *Inverse Problems*, **25**, no. 12, 123010, 59 pp., (2009).

[2] I.-K. Chen, H. Fujiwara and D. Kawagoe, Tomography from scattered signals obeying the stationary radiative transport equation, submitted.

[3] I. Chen, D. Kawagoe, Propagation of boundary-induced discontinuity in stationary radiative transfer and its application to the optical tomography, *Inverse Problems & Imaging* **13** no. 2, pp. 337–351, (2019).

*This work was supported by JSPS KAKENHI grant numbers JP20H01821, JP20K14344, and JP21H00999.
†Graduate School of Informatics, Kyoto University (d.kawagoe@acs.i.kyoto-u.ac.jp)