Initial-boundary value problem for a time-fractional diffusion system and an inverse problem of determining orders[†]

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Let $T \in \mathbb{R}_+ := (0, +\infty)$ and $\Omega \subset \mathbb{R}^d$ be a bounded domain with a sufficiently smooth boundary $\partial\Omega$ (e.g. of C^2 class). For a constant $K \in \mathbb{N}$, let $\alpha_1, \ldots, \alpha_K$ be constants satisfying $1 > \alpha_1 \geq \cdots \geq \alpha_K > 0$.
Consider the following initial-boundary value problem for a coupled system of time-fractional diffusion equations

$$\begin{cases} (\partial_t^{\boldsymbol{\alpha}} + \mathcal{A})\boldsymbol{u} = \mathcal{P}\boldsymbol{u} + \boldsymbol{F} & \text{in } \Omega \times (0,T), \\ \boldsymbol{u} = \boldsymbol{u}_0 & \text{in } \Omega \times \{0\}, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{1}$$

where $\boldsymbol{u}(\boldsymbol{x}) := (u_1(\boldsymbol{x}), \ldots, u_K(\boldsymbol{x}))^T$, $\boldsymbol{u}_0(\boldsymbol{x}) := (u_0^{(1)}(\boldsymbol{x}), \ldots, u_0^{(K)}(\boldsymbol{x}))^T$, $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_K)^T$, $\boldsymbol{F}(\boldsymbol{x}, t) := (F_1(\boldsymbol{x}, t), \ldots, F_K(\boldsymbol{x}, t))^T$, and $\partial_t^{\boldsymbol{\alpha}}\boldsymbol{u} := (\partial_t^{\alpha_1} u_1, \ldots, \partial_t^{\alpha_K} u_K)^T$. Moreover, $\mathcal{A} := \mathrm{diag}\{\mathcal{A}_1, \ldots, \mathcal{A}_K\}$ is a symmetric elliptic operator and the coupled operator $\mathcal{P}$ includes at most first-order derivatives (in space) of the solution.

We first prove the well-posedness of the above initial-boundary value problem with $\boldsymbol{u}_0 \in (L^2(\Omega))^K$ and $\boldsymbol{F} \in (L^p(0,T; L^2(\Omega)))^K$, $p \in [1, \infty]$.

After that, we restrict (1) to a weakly coupled system with t-independent coefficients, that is, $\mathcal{P}\boldsymbol{u} = \boldsymbol{C}\boldsymbol{u}$ where $\boldsymbol{C} := (c_{k\ell})_{1 \leq k, \ell \leq K}$ is a matrix-valued function. Then we show the long-time asymptotic estimate:

$$\|\boldsymbol{u}(\cdot, t)\|_{H^2(\Omega)} \leq C t^{-\alpha_K} \|\boldsymbol{u}_0\|_{L^2(\Omega)}, \quad \forall t >> 1.$$

Here the decay rate $t^{-\alpha_K}$ is sharp provided that $u_0^{(K)} \not\equiv 0$ in Ω.

Furthermore, let $d = 1, 2, 3$, we investigate the uniqueness result of the following inverse problem:

Order determination problem (by one compoent)
Let $\boldsymbol{u}$ satisfy (1) and fix $T > 0$, $\boldsymbol{x}_0 \in \Omega$, $k_0 \in \{1, \ldots, K\}$ arbitrarily. Determine the orders $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_K)^T$ of (1) by the single point observation of the k_0-th component u_{k_0} of $\boldsymbol{u}$ at $\{\boldsymbol{x}_0\} \times (0, T)$.

References

[1] Z. Li, X. Huang and Y. Liu, Initial-boundary value problems for coupled systems of time-fractional diffusion equations, Submitted. arXiv:2209.03767

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